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Optimal Angle Reduction - A Behavioral Approach to Linear System Approximation

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Abstract

We investigate the problem of optimal state reduction under minimization of the angle between system behaviors. The angle is defined in a worst-case sense, as the largest angle that can occur between a system trajectory and its optimal approximation in the reduced order model. This problem is analysed for linear time-invariant finite dimensional systems, in a behavioral ℓ_2-setting, without reference to input/output decompositions and stability considerations. The notion of a weakest past-future link is introduced and it is shown how this concept is applied for the purpose of model reduction. A method that reduces the state dimension by one is presented and shown to be optimal. Specific algorithms are provided for the numerical implementation of the approximation method.

Keywords

Optimal model reduction, State space balancing, ℓ_2-systems, Least squares optimization, Gap metrics, Hankel-norm reduction.

1 Introduction

The general aim of model approximation is to replace a complex dynamical system by a simpler, less complex one without undue loss of accuracy. Model approximation techniques have been proven to be of paramount interest in engineering and in areas where modeling, control and system identification are the key elements in the analysis and synthesis of dynamical systems. In econometrics and statistical data analysis, model approximation is commonly used to reduce the order of high order regression models. In identification and spectral estimation, high order estimates are often used as the basis for lower order approximants.

Many techniques have been developed for approximating a complex system by a simpler one. The standard paradigm is to approximate a linear time-invariant system of McMillan degree n by another linear time-invariant

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system of lower degree such that the behavior of the approximate system resembles that of the original, more complex system. Balanced truncations [8, 13, 18, 21], optimal Hankel norm reductions [1, 11, 17, 28], spectral reductions [16], Padé approximations [5], projection techniques [29] and model reductions by means of Akaike's canonical correlations [2, 15, 20] are common examples of this theme. See also [4, 12, 19, 22] for other seminal contributions in this field.

In many such theories, systems are assumed to be stable, in input-output form and often with stochastic assumptions on system variables. In spite of the unquestionable strength and widespread applications of these model approximation techniques this paradigm has, however, some important shortcomings.

Firstly, few techniques provide quantitative insight in the question of the accuracy of the approximate model with respect to the original, complex one. Many model reduction techniques are based on heuristic procedures and the quality of the approximate model is usually judged on the basis of visual inspection of typical system responses, such as frequency responses, impulse responses, etc. Obviously, the lack of a rigorous quality assessment of approximate models is unsatisfactory from a system theoretic point of view.

Secondly, many models of physical and economical systems do not allow an obvious or natural represention in input-output form. Assuming such an input-output structure to be present is undesirable for at least two reasons. The first is a pragmatic one: if a system has no obvious partitioning of input and output variables, there seems little reason to assume one for the sake of a paradigm. The second reason is related to the non-uniqueness of such a partitioning: different choices of input and output variables lead to different approximate models which seems undesirable. The effect of the non-uniqueness (in the choice of input and output variables) on approximate models has never been subject of investigation.

Thirdly, the fact that many model approximation techniques assume stability of the system, constitutes a severe limitation for many practical situations. For unstable systems it is in general undesirable to apply model approximation techniques on the stabilized system.

The present paper is motivated by these shortcomings. We investigate an optimal model approximation problem for the class of linear time-invariant systems on discrete time. Following the behavioral framework [32, 33], systems will be defined as sets of time series. It is a distinctive feature of our approach that system variables are treated in a symmetric way without an explicit distinction between input and output variables. The theoretical development is carried out without reference to system representations and without stability assumptions. Obviously, a theory on *optimal* model approximation should start with concise definitions of model classes, and notions such as system complexity and system accuracy. Roughly speaking, we address the question of synthesizing a linear time-invariant dynamical system whose state dimension is strictly smaller than the one of a given system, and such that the angle between the two systems is minimized. The angle between two systems is defined in a worst-case sense, as the largest angle that can occur between the trajectories in one system and their closest approximations in the other. Here, 'closest approximation' will be understood in a deterministic least squares sense.

The outline of the paper is as follows. Section 2 contains basic definitions and preliminaries, and Section 3 the formulation of the *optimal angle reduction* problem. In Section 4 we introduce the concept of canonical past-future links, and in Section 5 it is shown how truncation of these links generate optimal angle approximants, for systems of (co-)rank one and reduction of the degree by one. Partial results are obtained for reducing the degree by one for systems of arbitrary rank. Section 6 describes how this reduction technique is implemented in terms of canonical isometric state representations. Explicit reduction formulas are given, together with some error bounds for lower order approximations derived from sequential reductions. In Section 7 we compare the approach with some other reduction techniques, in particular with optimal Hankel norm approximation [11]. Section 8 contains an exact numerical example.

To make the paper as self contained as possible, we have incorporated many proofs of basic system properties. We believe that this enhances the tutorial value of the paper. Some early results of this work have been published in [26, 27].

Notation

Integers, positive integers, and the real and complex numbers are denoted by $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$, respectively. $\mathbb{Z}_+$ and $\mathbb{Z}_-$ denote the non-negative and negative elements of $\mathbb{Z}$, respectively. For $T \subseteq \mathbb{Z}$ and $(W, \|\cdot\|)$ a normed vector space we define $\ell(T, W) := W^T$ and $\ell_2(T, W) := \{w \in \ell(T, W) \mid \sum_{t \in T} \|\mathbf{w}_t\|^2 < \infty\}$. The latter space is equipped with its usual inner product, $\langle \cdot, \cdot \rangle$, and is also denoted as ℓ_2 or ℓ_2^q if the dimension q of W is relevant for the context. Further, $\ell_2^- := \ell_2(\mathbb{Z}_-, W)$ and $\ell_2^+ := \ell_2(\mathbb{Z}_+, W)$. The symbol $\mathbf{0}$ will indicate the zero-element in any of these sets. The evaluation of a time series $\mathbf{w} \in \ell(T, W)$ at time $t \in T$ will be denoted as $\mathbf{w}_t$. Multiple evaluations will be denoted as $\{\ldots \mathbf{w}_{-2}, \mathbf{w}_{-1} | \mathbf{w}_0, \mathbf{w}_1 \ldots\}$ where the symbol $|$ is used to separate evaluations in $\mathbb{Z}_-$ and $\mathbb{Z}_+$. The restriction of a time series $\mathbf{w}$ and the restriction of a set of time series $\mathfrak{B} \subset \ell(\mathbb{Z}, W)$ to a subset $\mathcal{I} \subset \mathbb{Z}$ is denoted as $\mathbf{w}_{\mathcal{I}}$, and $\mathfrak{B}_{\mathcal{I}}$. For $k \in \mathbb{Z}$, $\sigma^k : \ell_2 \to \ell_2$ denotes the *k-shift* $(\sigma^k \mathbf{w})_t = \mathbf{w}_{t+k}$. We refer to *left-shifts* if $k > 0$ and to *right-shifts* if $k < 0$. Left-shifts are also applied to ℓ_2^+ and right-shifts to ℓ_2^-, with obvious definitions. If $\mathbf{w} \in \ell_2$ is a multivariate time series, then shifts($\mathbf{w}$) denotes the collection of all shifts of $\mathbf{w}$, i.e., shifts($\mathbf{w}$) $:= \{\sigma^k \mathbf{w} \mid k \in \mathbb{Z}\}$. The symbol $\perp$ is defined, given $\mathbf{w}, \mathbf{w}' \in \ell_2$, as $\mathbf{w} \perp \mathbf{w}' :\Leftrightarrow \langle \mathbf{w}, \mathbf{w}' \rangle = 0$. If $\mathfrak{B}, \mathfrak{B}' \subseteq \ell_2$ then $\mathfrak{B} \perp \mathfrak{B}'$ means that $\mathbf{w} \perp \mathbf{w}'$ for all $\mathbf{w} \in \mathfrak{B}$ and $\mathbf{w}' \in \mathfrak{B}'$. The symbol $\underset{t}{\wedge}$ denotes the concatenation product of time series at time t, i.e. $\mathbf{w} \underset{t}{\wedge} \mathbf{w}'$ denotes the time series $\{\ldots, \mathbf{w}_{t-2}, \mathbf{w}_{t-1}, \mathbf{w}'_t, \mathbf{w}'_{t+1}, \ldots\}$. We write $\wedge$ for $\underset{t}{\wedge}$ if the concatenation instant t is obvious from the context. The k-th unit pulse in ℓ_2^q is denoted as $\mathbf{e}_k$ and defined as the time series which is equal to zero except for a unit entry at the k-th component at time $t = 0$. For $n \in \mathbb{N}$, I_n is the $n \times n$ identity matrix, e_k is the k-th standard unit vector of $\mathbb{R}^n$, and $E_k \in \mathbb{R}^{n \times (n-1)}$ is the matrix I_n from which the k-th column has been removed.

2 Systems

2.1 Dynamical systems

Following the behavioral framework initiated by Willems [32, 33], a dynamical system, or a *system* for short, is a set of mappings $\mathbf{w} : T \to W$ defined on a *time set* T and taking values in a *signal space* W. A system is therefore a subset $\mathfrak{B} \subseteq \ell(T, W)$. Elements in $\mathfrak{B}$ are called *trajectories* and we sometimes refer to $\mathfrak{B}$ as the *behavior*. In this paper we exclusively consider systems with discrete time set $T = \mathbb{Z}$ and finite dimensional real-valued signal spaces $W = \mathbb{R}^q$. We will further focus on ℓ_2-systems which are subsets of ℓ_2. A system $\mathfrak{B}$ is called *linear* if $\mathfrak{B}$ is a linear subspace of $\ell(T, W)$. It is called *time-invariant* if $\mathbf{w} \in \mathfrak{B}$ implies that the k-shifted trajectory $\sigma^k \mathbf{w}$ belongs to $\mathfrak{B}$ for any integer $k \in \mathbb{Z}$. Further, a subset $\mathfrak{B} \subseteq \ell_2$ is called *ℓ_2-complete* if a trajectory $\mathbf{w}$ belongs to $\mathfrak{B}$ whenever $\mathbf{w} \in \ell_2$ and its restrictions $\mathbf{w}_{\mathcal{I}} \in \mathfrak{B}_{\mathcal{I}}$ for all (finite) intervals $\mathcal{I} \subset \mathbb{Z}$. Linearity, time-invariance and ℓ_2-completeness define the system class which we study in this paper.

Definition 2.1 (System Class)
The system class $\mathbb{L}^q$ is the set of all linear, time-invariant and ℓ_2-complete systems $\mathfrak{B} \subseteq \ell_2(\mathbb{Z}, \mathbb{R}^q)$. We write $\mathbb{L}$ for $\mathbb{L}^q$ if the dimension q of the signal space is clear from the context.

Some first qualitative properties of systems in $\mathbb{L}$ are given in the following lemma.

Lemma 2.2
Systems in $\mathbb{L}$ are closed in the sense that they define closed subspaces of ℓ_2. All nonzero systems in $\mathbb{L}$ have infinite dimension.

Proof. To see the first statement, let $\mathfrak{B} \in \mathbb{L}$ and let $\mathbf{w}^{(i)} \in \mathfrak{B}$, $i \in \mathbb{Z}_+$ be a sequence of time series that converges (in the ℓ_2 topology) to $\mathbf{w} \in \ell_2$. Let $\mathcal{I} \subset \mathbb{Z}$ be an interval. Then $\mathbf{w}^{(i)}_{\mathcal{I}} \in \mathfrak{B}_{\mathcal{I}}$ and since $\mathfrak{B}_{\mathcal{I}}$ is finite dimensional, the limit $\mathbf{v}_{\mathcal{I}} := \lim_{i\to\infty} \mathbf{w}^{(i)}_{\mathcal{I}}$ exists, is contained in $\mathfrak{B}_{\mathcal{I}}$ and $\mathbf{v}_{\mathcal{I}} = \mathbf{w}_{\mathcal{I}}$. Hence $\mathbf{w}_{\mathcal{I}} \in \mathfrak{B}_{\mathcal{I}}$ for all intervals $\mathcal{I} \subset \mathbb{Z}$. Since $\mathbf{w} \in \ell_2$ and $\mathfrak{B}$ is ℓ_2-complete, it follows that $\mathbf{w} \in \mathfrak{B}$. Conclude that $\mathfrak{B}$ is closed. To prove the second statement, suppose that $\mathfrak{B}$ has finite dimension n. This implies that for every $\mathbf{w} \in \mathfrak{B}$ the elements $\mathbf{w}, \sigma\mathbf{w}, \ldots, \sigma^n\mathbf{w}$ are linearly dependent. Hence $\mathbf{w}$ satisfies an equation of the form $\alpha_0\mathbf{w}_t + \ldots \alpha_n\mathbf{w}_{t+n} = 0$ where $\alpha_i \in \mathbb{R}$. Each component of $\mathbf{w}$ is therefore a polynomial-exponential time series. However, the only square summable polynomial-exponential time series is $\mathbf{0}$. Hence, $\mathfrak{B} = \mathbf{0}$. □

We remark that not every linear closed and time-invariant subspace of ℓ_2 belongs to $\mathbb{L}$. See Example 2.9 below.

Example 2.3
The system

$$\mathfrak{D} = \{\mathbf{w} = \left(\begin{smallmatrix}\mathbf{u}\\ \mathbf{y}\end{smallmatrix}\right) \in \ell_2^2 \mid \mathbf{y}_t = \mathbf{u}_t - \mathbf{u}_{t-1}\}. \tag{2.1}$$

will be used as a simple illustration of notions throughout. This is a system in $\mathbb{L}^2$. It models 'taking first differences', as the second component is the first difference of the first component.

2.2 Rank and degree

The complexity of a system is a measure of how many trajectories the system allows. The dimension of a system restricted to a (finite) time-window is taken as a useful measure of system complexity. The following definition exploits the fact that for systems in the model class $\mathbb{L}$ this dimension is an affine function of the length of the time-window.

Definition 2.4 (Rank, degree, complexity)
The complexity *of a dynamical system $\mathfrak{B} \in \mathbb{L}^q$ is the pair of integers (m, n) such that* $\dim(\mathfrak{B}_{[0,N-1]}) = mN + n$ *for all $N \geq n$. The number m is called the* rank *of the system, $q - m$ its* co-rank, *and n its* degree.

Proof of correctness. A proof is given in [23] and based on results in [32]. As the argument is also needed in other proofs, we give a (slightly adapted) proof. Let $\mathfrak{B} \in \mathbb{L}^q$ and consider the integers $L_k := \dim\{w \in \mathbb{R}^q | \mathbf{0} \wedge w \in \mathfrak{B}_{[-k,0]}\}$. Then $\dim(\mathfrak{B}_{[0,N-1]}) = \Sigma_{k=0}^{N-1} L_j$ and the sequence $\{L_k\}$ is non-increasing and bounded by q. Consequently, the limit $L^* := \lim_{k\to\infty} L_k$ exists and is achieved for some finite $k^* \in \mathbb{Z}_+$. This implies that for all $N \geq k^*$, $\dim(\mathfrak{B}_{[0,N-1]}) = mN + n$ with $m = L^*$ and $n = \Sigma_{k=0}^{k^*-1}(L_k - L^*)$. Finally, all terms in the latter summation are at least one, so $k^* \leq n$ and hence the dimension formula is valid for all $N \geq n$. □

Lemma 2.5
Let $\mathfrak{B} \in \mathbb{L}^q$ and let (m, n) denote its complexity. Then

1. *$0 \leq m \leq q$. If $m = 0$ then $n = 0$ and $\mathfrak{B} = \mathbf{0}$; if $m = q$, then $n = 0$ and $\mathfrak{B} = \ell_2^q$.*
2. $\dim\{\mathbf{w} \in \mathfrak{B}_{[0,N-1]} \mid \mathbf{0} \underset{0}{\wedge} \mathbf{w} \in \mathfrak{B}_{(-\infty,N-1]}\} = \dim\{\mathbf{w} \in \mathfrak{B}_{[0,N-1]} \mid \mathbf{w} \underset{N}{\wedge} \mathbf{0} \in \mathfrak{B}_{[0,\infty)}\} = mN.$

3. If $m \neq 0$ then $\mathfrak{B}$ contains a non-zero trajectory of finite support.

PROOF. 1. As $0 \leq \dim \mathfrak{B}_{[0,N]} - \dim \mathfrak{B}_{[0,N-1]} \leq q$. If $m = 0$ then $\dim \mathfrak{B}_{[0,N-1]} = n$ for all $N \geq n$, and hence $\dim \mathfrak{B} = n$. It follows from Lemma 2.2 that $\mathfrak{B} = \mathbf{0}$, and hence $n = 0$. If $m = q$, then $\mathfrak{B}_{[0,N-1]} = \ell_2([0, N-1], \mathbb{R}^q)$, and completeness implies $\mathfrak{B} = \ell_2^q$, and $n = 0$.

2. Let $\mathcal{I} \subset \mathbb{Z}$ be an interval of length N and define $\mathfrak{B}_{\mathcal{I}}^{0\wedge}$ as the subspace of $\mathfrak{B}_{\mathcal{I}}$ whose trajectories can be preceded by zeros in $\mathfrak{B}$. It follows from the definition of $L^*(= m)$ in the proof of correctness of Definition 2.4 that $\mathfrak{B}_{\mathcal{I}}^{0\wedge}$ has dimension mN. Similarly, the dimension of $\mathfrak{B}_{\mathcal{I}}^{\wedge 0}$, the subspace of $\mathfrak{B}_{\mathcal{I}}$ whose trajectories can be followed by zeros, also has dimension mN.

3. Let $\mathcal{I} \subset \mathbb{Z}$ be an interval of length N and define $\mathfrak{B}_{\mathcal{I}}^{0} := \{\mathbf{w} \in \mathfrak{B}_{\mathcal{I}} \mid \mathbf{0} \wedge \mathbf{w} \wedge \mathbf{0} \in \mathfrak{B}\}$. Then

$$\mathfrak{B}_{\mathcal{I}}^{0} = \mathfrak{B}_{\mathcal{I}}^{0\wedge} \cap \mathfrak{B}_{\mathcal{I}}^{\wedge 0} \tag{2.2}$$

and both sets in the righthand side are mN-dimensional subspaces of $\mathfrak{B}_{\mathcal{I}}$, which itself has dimension $mN + n$ for $N \geq n$. Hence

$$mN - n \leq \dim \mathfrak{B}_{\mathcal{I}}^{0} \leq mN \tag{2.3}$$

for all $N \geq n$. Since the lower bound is strictly positive if $m \neq 0$ and $N > n$, we conclude that $\mathfrak{B}$ contains a non-zero trajectory of finite support. □

It follows that all ℓ_2-systems in one variable (i.e. with $q = 1$) are trivial. The rank of a system denotes its degree of freedom at each time instant. The degree corresponds to the degree of freedom due to initial conditions. These numbers have elegant system theoretic interpretations. In fact, the rank and the degree are, resp., the dimensions of the input- and state space of any (minimal) input-state-output representation of the system. See section 6. Further, it can be shown that a system $\mathfrak{B} \in \mathbb{L}^q$ of rank m and degree n is the set of ℓ_2 solutions of $q - m$ ordinary difference equations of total order n.

EXAMPLE 2.6
Consider the system $\mathfrak{D}$ given by (2.1). At any finite interval $\mathcal{I}$ of length N either $\mathbf{u}$ or $\mathbf{y}$ can be given arbitrary values, and one initial or end value of the other variable is arbitrary. This implies that $\dim \mathfrak{D}_{\mathcal{I}} = N + 1$, so the rank and degree of $\mathfrak{D}$ are both one.

2.3 Orthogonal Complement

The orthogonal complement of an ℓ_2-system $\mathfrak{B}$ is defined as the set of square summable time series that are orthogonal to all elements of the system, i.e.,

$$\widetilde{\mathfrak{B}} := \mathfrak{B}^{\perp} := \{\widetilde{\mathbf{w}} \in \ell_2^q \mid \mathbf{w} \perp \widetilde{\mathbf{w}} \text{ for all } \mathbf{w} \in \mathfrak{B}\}.$$

Some basic properties of the orthogonal complement are summarized in the following proposition.

PROPOSITION 2.7 (ORTHOGONAL COMPLEMENT)
Let $\mathfrak{B}$ be a system in $\mathbb{L}^q$ of complexity (m, n) and let $\widetilde{\mathfrak{B}}$ be its orthogonal complement. Then

1. $\ell_2^q = \mathfrak{B} \oplus \widetilde{\mathfrak{B}}$ and $(\widetilde{\mathfrak{B}})^{\perp} = \mathfrak{B}$.

2. $\widetilde{\mathfrak{B}}$ *belongs to* $\mathbb{L}^q$.

3. $\widetilde{\mathfrak{B}}$ *has complexity* $(q-m, n)$.

PROOF. 1. This is a well-known property of closed subspaces of Hilbert spaces, cf. Lemma 2.2

2. Clearly, $\widetilde{\mathfrak{B}}$ is linear and shift-invariant. ℓ_2-completeness can be deduced from Statement 3 of Lemma 2.5.

3. Let $\mathcal{I} \subset \mathbb{Z}$ be an interval of length N and observe that $(\widetilde{\mathfrak{B}}_{\mathcal{I}})^{\perp} = \mathfrak{B}^0_{\mathcal{I}}$ (see (2.2) for the notation), as indeed $\bar{\mathbf{w}} \perp \widetilde{\mathfrak{B}}_{\mathcal{I}}$ if and only if $\mathbf{0} \wedge \bar{\mathbf{w}} \wedge \mathbf{0}$ in $\mathfrak{B}$. So $\dim \widetilde{\mathfrak{B}}_{\mathcal{I}} + \dim \mathfrak{B}^0_{\mathcal{I}} = qN$. From (2.3) it follows that $(q-m)N \leq \dim \widetilde{\mathfrak{B}}_{\mathcal{I}} \leq (q-m)N + n$ for all $N \geq n$. Conclude that $\widetilde{\mathfrak{B}}$ has rank $q-m$ and degree $\tilde{n} \leq n$. From $(\widetilde{\mathfrak{B}})^{\perp} = \mathfrak{B}$ it follows that $n \leq \tilde{n}$, so $n = \tilde{n}$. □

The reader may skip the remainder of Section 2 and refer to this part when it is required in Section 5.

2.4 From trajectory to system

Dynamical systems can be generated from a finite number of time series by a process called *completion*. For a subspace $\mathfrak{B} \subseteq \ell_2^q$ its completion is defined as

$$\mathrm{comp}(\mathfrak{B}) := \{\mathbf{w} \in \ell_2^q \mid \mathbf{w}_{\mathcal{I}} \in \mathfrak{B}_{\mathcal{I}} \text{ for all (finite) intervals } \mathcal{I} \subset \mathbb{Z}\}.$$

It follows that $\mathrm{comp}(\mathfrak{B})$ is the smallest ℓ_2-complete set that contains $\mathfrak{B}$. Note that $\mathfrak{B} \in \mathbb{L}^q$ implies that $\mathrm{comp}(\mathfrak{B}) = \mathfrak{B}$.

DEFINITION 2.8 (SYSTEM GENERATED BY TRAJECTORIES)
The system generated by a finite set of time series $\mathbf{w}_{(i)} \in \ell_2^q$, $i = 1, \ldots, m$ *is defined as*

$$\mathfrak{B}(\mathbf{w}_{(1)}, \ldots, \mathbf{w}_{(m)}) = \mathrm{comp}(\mathrm{span}[\mathrm{shifts}(\mathbf{w}_{(1)}, \ldots, \mathbf{w}_{(m)})]).$$

Note that $\mathfrak{B}(\mathbf{w}_{(1)}, \ldots, \mathbf{w}_{(m)})$ belongs to $\mathbb{L}^q$, as the completion process does not distort linearity and time-invariance. Obviously $\mathfrak{B}(\mathbf{w}_{(1)}, \ldots, \mathbf{w}_{(m)})$ is the smallest dynamical system (in $\mathbb{L}^q$) containing $\{\mathbf{w}_{(1)}, \ldots, \mathbf{w}_{(m)}\}$. In view of concepts introduced in [32], it is also called the *most powerful unfalsified model* of the given time series.

EXAMPLE 2.9
The completion process may considerably enlarge a set of time-series. An example taken from [14] illustrates this. Let

$$\mathbf{w} := \left[\begin{array}{ccc|ccc} \cdots & 0 & 0 & 1 & 0 & 0 \cdots \\ \cdots & 0 & 0 & \frac{1}{1^2} & \frac{1}{2^2} & \frac{1}{3^2} \cdots \end{array}\right].$$

Every finite time series can be decomposed in a basis consisting of shifts of $\mathbf{w}$, i.e. $\{\mathrm{span}[\mathrm{shifts}(\mathbf{w})]\}_{\mathcal{I}} = \mathbb{R}^{2\times N}$ where N is the length of $\mathcal{I}$. Hence $\mathfrak{B}(\mathbf{w}) = \ell_2$. This also shows that $\mathrm{span}[\mathrm{shifts}(\mathbf{w})]$ is a time-invariant, linear closed subset of ℓ_2 that is not ℓ_2-complete. In fact, a generic time series (suitably defined) in ℓ_2^q generates the *trivial* system ℓ_2^q.

The example shows that time series may be "too rich" to qualify as generators of systems of finite complexity. We will therefore consider systems that are generated by time series of finite degree. Formally, a time series $\mathbf{w} \in \ell_2$ is said to have *finite degree* if both its *forward degree* and its *backward degree* are finite. Here,

$$\begin{aligned} \text{forward degree} &:= \dim(\operatorname{span}\{(\sigma^{j+1}\mathbf{w})_{\mathbb{Z}_+}\}_{j\in\mathbb{Z}_+}) \\ \text{backward degree} &:= \dim(\operatorname{span}\{(\sigma^{-j}\mathbf{w})_{\mathbb{Z}_-}\}_{j\in\mathbb{Z}_+}). \end{aligned} \tag{2.4}$$

The *total degree* of a finite set of finite degree time series is the sum of the forward degrees and backward degrees of their elements. Well known examples of finite degree trajectories are impulse- and step responses of finite dimensional linear time-invariant systems in input-output form.

Proposition 2.10 (Finite degree generators)
Let $\mathbf{w}_{(1)}, \dots, \mathbf{w}_{(m)} \in \ell_2^q$ be a set of finite degree trajectories with total degree n. Then $\mathfrak{B} := \mathfrak{B}(\mathbf{w}_{(1)}, \dots, \mathbf{w}_{(m)})$ has rank at most m and degree at most n.

Proof. Let $\mathbf{w}_{(1)}, \dots, \mathbf{w}_{(m)}$ have finite degree en let $\mathcal{I} = [0, N-1]$, $N > 0$. We first show that

$$\mathfrak{B}_{\mathcal{I}} = \big(\operatorname{span}[\operatorname{shifts}(\mathbf{w}_{(1)}, \dots, \mathbf{w}_{(m)})]\big)_{\mathcal{I}}. \tag{2.5}$$

To see this, consider the spaces spanned by shifts over a restricted lag,

$$\mathfrak{B}_{\mathcal{I}}^{(K)} := \Big(\operatorname{span}[\sigma^{i_1}\mathbf{w}_{(1)}, \dots \sigma^{i_m}\mathbf{w}_{(m)}]_{-K\le i_j\le K}\Big)_{\mathcal{I}}. \tag{2.6}$$

Then

$$0 \subseteq \mathfrak{B}_{\mathcal{I}}^{(0)} \subseteq \dots \subseteq \mathfrak{B}_{\mathcal{I}}^{(K)} \subseteq \mathfrak{B}_{\mathcal{I}}^{(K+1)} \dots$$

defines a nested sequence of linear subspaces in $\mathbb{R}^{qN}$. Hence its limit $\lim_{K\to\infty} \mathfrak{B}_{\mathcal{I}}^{(K)}$ is attained for finite K, and this limit must be $\mathfrak{B}_{\mathcal{I}}$. Finally, let n' denote the sum of all forward and backward degrees. All shifts in the definition of $\mathfrak{B}_{\mathcal{I}}^{(K)}$ in (2.6) with $i_j \notin [-(N-1), 0]$ contribute at most n' to the dimension of the behaviour restricted to $\mathcal{I}$, for all K. Hence $\dim \mathfrak{B}_{\mathcal{I}} \le mN + n'$. □

Remark 2.11
As an alternative to Definition 2.8, systems can be generated from trajectories by means of *convolutions*. Let $\mathbf{w} \in \ell_2^q$ and define

$$\mathfrak{B}'(\mathbf{w}) := \{\mathbf{v} * (\sigma^j \mathbf{w}) \mid \mathbf{v} \in \ell_2(\mathbb{Z}, \mathbb{R}),\ j \in \mathbb{Z}\} \tag{2.7}$$

where $*$ denotes convolution. Then, generically, $\mathfrak{B}(\mathbf{w}) = \mathfrak{B}'(\mathbf{w})$ but $\mathfrak{B}'(\mathbf{w})$ may not yield an ℓ_2-complete system. For instance, let $\mathbf{w} = \mathbf{b} + \sigma\mathbf{b}$ with $\mathbf{b}$ a non-zero element in ℓ_2^q. Then $\mathbf{b} \in \mathfrak{B}(\mathbf{w})$, but $\mathbf{b} \notin \mathfrak{B}'(\mathbf{w})$.

2.5 From system to trajectory

Next, we discuss the reverse question of how to obtain a set of generating trajectories from a given dynamical system.

Proposition 2.12 (Trajectories generating a given system)
Let $\mathfrak{B} \in \mathbb{L}^q$ be a system with rank m and degree n.

1. *There exist m time series of finite degree $\mathbf{w}_{(1)}, \dots, \mathbf{w}_{(m)} \in \ell_2^q$ that generate $\mathfrak{B}$, and $\mathfrak{B}$ cannot be generated by fewer time series of finite degree.*
2. *The minimum total degree of any set of time series that generates $\mathfrak{B}$ is n.*
3. *If $m = 1$ then $\mathfrak{B}(\mathbf{w}) = \mathfrak{B}$ for all non-zero $\mathbf{w} \in \mathfrak{B}$.*

PROOF. 1. Let $\mathfrak{B} \in \mathbb{L}$ have rank m and consider the induction hypothesis that $\mathfrak{B}$ can be generated by m finite degree trajectories. If $m = 0$, then by Lemma 2.5, $\mathfrak{B} = \mathbf{0}$ and the hypothesis is correct. If $m > 0$ then $\mathfrak{B}$ contains a non-zero trajectory, say $\mathbf{w}$, with finite support (cf. Lemma 2.5). Then $\mathbf{w}$ has finite degree and we define $\mathbf{w}_{(m)} := \mathbf{w}$. Consider $\mathfrak{B} \cap \mathfrak{B}(\mathbf{w})^{\perp}$. This set belongs to $\mathbb{L}^q$, has rank $m-1$, and by induction hypothesis there are $m-1$ trajectories of finite degree that generate this subsystem, and hence, together with $\mathbf{w}_{(m)}$, generate $\mathfrak{B}$.

2. The lower bound follows from Proposition 2.10 and $\dim(\mathfrak{B}_{[0,N-1]}) = mN + n$. Further, a set of m time series of total degree n is a so called *minimum lag* description in terms of difference equations for the orthogonal complement of $\mathfrak{B}$. Existence of such representations is proved in e.g. [33, Proposition X.5].

3. Clearly $\mathfrak{B}(\mathbf{w}) \subset \mathfrak{B}$ for all $\mathbf{w} \in \mathfrak{B}$, and for $\mathbf{w} \neq \mathbf{0}$ both systems are of rank one. If this is a strict inclusion for $\mathbf{w} \neq 0$, this would imply that $\{\mathbf{w}' \in \mathfrak{B} \mid \mathbf{w}' \perp \mathfrak{B}(\mathbf{w})\}$ is a finite dimensional linear shift-invariant subspace of ℓ_2^q. By Lemma 2.5, this set is $\mathbf{0}$ which shows that $\mathfrak{B}(\mathbf{w}) = \mathfrak{B}$, as desired. □

From statement 1 of Proposition 2.12 we infer that every system in $\mathbb{L}$ can be generated by a finite number of finite degree trajectories. Proposition 2.12 shows that the notion of rank (Definition 2.4) is a straightforward generalization of the rank of matrices. The image of a matrix of rank m has an m-dimensional basis. Likewise, a system of rank m is generated by m time series. The degree determines the time-span of the system dynamics.

EXAMPLE 2.13
Consider the trajectory in $\mathfrak{D}$ given by

$$\mathbf{w} = \left[\begin{array}{ccc|ccc} \dots & 0 & 0 & 1 & 0 & \dots \\ \dots & 0 & 0 & 1 & -1 & \dots \end{array} \right].$$

It is of finite degree, as the sets in (2.4) have dimensions 0 and 1. It generates the system $\mathfrak{D}$ in the sense that $\mathfrak{B}(\mathbf{w}) = \mathfrak{D}$. Also $\mathbf{w}' := \mathbf{w} + \sigma\mathbf{w}$ generates $\mathfrak{D}$, while $\mathfrak{D} \neq \{\mathbf{v} * (\sigma^j \mathbf{w}') \mid \mathbf{v} \in \ell_2, j \in \mathbb{Z}\}$, as the latter space does not contain $\mathbf{w}$, cf. (2.7).

3 The model approximation problem

In the first part of this section a measure of 'distance' between two systems in $\mathbb{L}$ is introduced. This leads to the formulation of a model approximation problem in the second part of this section.

3.1 Angle between systems

As an approximation criterion for dynamical systems we consider the *angle* between two systems. This is defined as follows. The angle between two square summable time series is given by

$$\theta(\mathbf{w},\mathbf{w}') := \begin{cases} 0 & \text{if } \mathbf{w}=\mathbf{0} \text{ and } \mathbf{w}'=\mathbf{0} \\ \arccos\left(\frac{|\langle \mathbf{w},\mathbf{w}'\rangle|}{\|\mathbf{w}\|\|\mathbf{w}'\|}\right) & \text{if } \mathbf{w}\neq\mathbf{0} \text{ and } \mathbf{w}'\neq\mathbf{0} \\ \pi/2 & \text{if either } \mathbf{w}=\mathbf{0} \text{ or } \mathbf{w}'=\mathbf{0} \end{cases}.$$

This definition is motivated by the geometric analog of orthogonal projections in finite dimensional vector spaces. The angle between a time series and a closed linear subspace $\mathfrak{B}' \subseteq \ell_2$ is defined as the minimal possible angle between the time series and elements of $\mathfrak{B}'$, i.e.,

$$\theta(\mathbf{w},\mathfrak{B}') := \min_{\mathbf{w}'\in\mathfrak{B}'} \theta(\mathbf{w},\mathbf{w}').$$

This minimum exists and it is easy to see that it is attained for the orthogonal projection of $\mathbf{w}$ onto $\mathfrak{B}'$, i.e. $\theta(\mathbf{w},\mathfrak{B}') = \theta(\mathbf{w},\mathbf{w}')$ with $\mathbf{w}'$ the orthogonal projection of $\mathbf{w}$ onto $\mathfrak{B}'$. The question how to compute projections is deferred to section 6. Figure 1 depicts the standard idea.

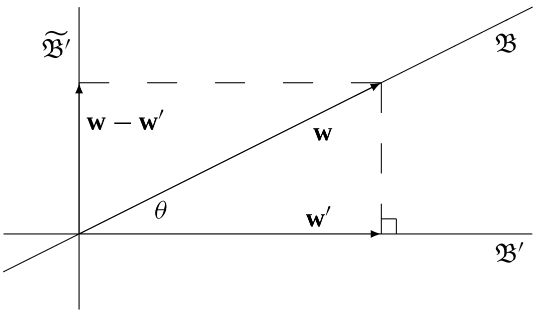

Figure 1: Orthogonal projection of $\mathbf{w}\in\mathfrak{B}$ onto $\mathfrak{B}'$, with $\alpha = \theta(\mathbf{w},\mathfrak{B}')$.

The angle between two systems is defined as the maximum angle that can occur between one system and elements of the other.

DEFINITION 3.1 (ANGLE)
The angle between two systems $\mathfrak{B}$ and $\mathfrak{B}'$ in $\mathbb{L}^q$ is defined as

$$\theta(\mathfrak{B},\mathfrak{B}') := \max\left\{\sup_{\mathbf{w}\in\mathfrak{B}} \theta(\mathbf{w},\mathfrak{B}'),\ \sup_{\mathbf{w}'\in\mathfrak{B}'} \theta(\mathbf{w}',\mathfrak{B})\right\}.$$

The angle is called flat *if $\theta(\mathfrak{B},\mathfrak{B}') = \theta(\mathbf{w},\mathfrak{B}') = \theta(\mathbf{w}',\mathfrak{B})$ for all nonzero $\mathbf{w}\in\mathfrak{B}$ and $\mathbf{w}'\in\mathfrak{B}'$*

The angles $\theta(\mathbf{w},\mathfrak{B}')$ and $\theta(\mathbf{w}',\mathfrak{B})$ are well defined and bounded by $\pi/2$ as systems in $\mathbb{L}$ define closed subspaces. This implies that the suprema are finite and hence $0 \leq \theta(\mathfrak{B},\mathfrak{B}') \leq \pi/2$. The angle is a metric on $\mathbb{L}^q$ as it is nonnegative, only zero if the systems are equal, symmetric in the arguments, and it satisfies the triangular inequality (See [6, 7]). We remark that the sinus of the angle corresponds to the gap between the two closed subspaces $\mathfrak{B}$ and $\mathfrak{B}'$, i.e., $\sin\theta(\mathfrak{B},\mathfrak{B}') = \mathrm{gap}(\mathfrak{B},\mathfrak{B}') := \|\Pi_{\mathfrak{B}} - \Pi_{\mathfrak{B}'}\|$ where $\Pi_{\mathfrak{B}}$ denotes the orthogonal projection of ℓ_2 onto the closed subspace $\mathfrak{B} \subseteq \ell_2$. See [6, 7, 9, 10] for details on the gap metric. Some other basic properties of the angle are summarized in the following lemma.

Lemma 3.2
Let $\mathfrak{B}$ and $\mathfrak{B}'$ be elements of $\mathbb{L}$ of rank m and m', respectively. Then

1. $\theta(\mathfrak{B},\mathfrak{B}') = \pi/2$ *if* $m \neq m'$.
2. *if* $\theta(\mathfrak{B},\mathfrak{B}') < \pi/2$ *then* $\sup_{\mathbf{w}\in\mathfrak{B}} \theta(\mathbf{w},\mathfrak{B}') = \sup_{\mathbf{w}'\in\mathfrak{B}'} \theta(\mathbf{w}',\mathfrak{B})$.
3. $\theta(\mathfrak{B},\mathfrak{B}') = \theta(\widetilde{\mathfrak{B}},\widetilde{\mathfrak{B}}')$.
4. $\theta(\mathfrak{B},\mathfrak{B}') + \theta(\widetilde{\mathfrak{B}},\mathfrak{B}') \geq \pi/2$ *and equality holds if and only if the angle is flat.*

Proof. To prove statement 1, let $\mathbf{w}_{(1)},\ldots,\mathbf{w}_{(m)}$ and $\mathbf{w}'_{(1)},\ldots,\mathbf{w}'_{(m')}$ be such that $\mathfrak{B} = \mathfrak{B}(\mathbf{w}_{(1)},\ldots,\mathbf{w}_{(m)})$ and $\mathfrak{B}' = \mathfrak{B}(\mathbf{w}'_{(1)},\ldots,\mathbf{w}'_{(m')})$ and suppose that $m < m'$. Let $\widehat{\mathbf{w}}_{(i)} := \Pi_{\mathfrak{B}'}\mathbf{w}_{(i)}$, $i = 1,\ldots,m$ and consider the system $\mathfrak{B}'' := \mathfrak{B}(\widehat{\mathbf{w}}_{(1)},\ldots,\widehat{\mathbf{w}}_{(m)})$. Since, $\widehat{\mathbf{w}}_{(i)} \in \mathfrak{B}'$ for all i, it follows that $\mathfrak{B}'' \subseteq \mathfrak{B}'$. If $m < m'$, then by Proposition 2.12, the latter inclusion is a proper one and hence there exists a non-zero $\mathbf{w}' \in \mathfrak{B}'$ which is orthogonal to all trajectories in $\mathfrak{B}$. Therefore, $\Pi_{\mathfrak{B}}\mathbf{w}' = \mathbf{0}$ so that

$$\sup_{\mathbf{w}\in\mathfrak{B}'} \theta(\mathbf{w},\mathfrak{B}) \geq \theta(\mathbf{w}',\mathfrak{B}) = \theta(\mathbf{w}',\mathbf{0}) = \pi/2.$$

Consequently, $\theta(\mathfrak{B},\mathfrak{B}') = \pi/2$. This also proves statement 3 in case $\mathfrak{B}$ and $\mathfrak{B}'$ are of different rank. If the ranks are the same, we prove that $\theta(\mathfrak{B},\mathfrak{B}') \leq \theta(\mathfrak{B}^{\perp},\mathfrak{B}'^{\perp})$, so that equality follows by a similar statement with $\mathfrak{B}$ and $\widetilde{\mathfrak{B}}$ interchanged. Let $\widehat{\mathbf{w}} \in \mathfrak{B}$ have norm 1. Then $\widehat{\mathbf{w}} = \gamma\widehat{\mathbf{w}}' + \widetilde{\gamma}\widetilde{\mathbf{w}}'$, with $\widehat{\mathbf{w}}' \in \mathfrak{B}'$ and $\widetilde{\mathbf{w}}' \in \widetilde{\mathfrak{B}}'$, with $\|\widehat{\mathbf{w}}'\| = \|\widetilde{\mathbf{w}}'\| = 1$ and $\gamma^2 + \widetilde{\gamma}^2 = 1$. So $\theta(\widehat{\mathbf{w}},\mathfrak{B}') = \theta(\widehat{\mathbf{w}},\widehat{\mathbf{w}}') = \arcsin(\widetilde{\gamma})$, and hence $\widetilde{\mathbf{w}}' = \gamma\widehat{\mathbf{w}} + \widetilde{\gamma}\bar{\mathbf{w}}$ for some $\bar{\mathbf{w}}$ with $\bar{\mathbf{w}} \perp \widehat{\mathbf{w}}$ and $\|\bar{\mathbf{w}}\| = 1$. Substituting this in the first equation gives $\widehat{\mathbf{w}} = \gamma^2\widehat{\mathbf{w}} + \gamma\widetilde{\gamma}\bar{\mathbf{w}} + \widetilde{\gamma}\widetilde{\mathbf{w}}'$ or $\widetilde{\mathbf{w}}' = \widetilde{\gamma}\widehat{\mathbf{w}} + \gamma\bar{\mathbf{w}}$. Hence $\theta(\widetilde{\mathbf{w}}',\mathfrak{B}) \leq \arcsin(\gamma)$ and it follows that $\theta(\widetilde{\mathfrak{B}},\widetilde{\mathfrak{B}}') \geq \theta(\widetilde{\mathbf{w}}',\widetilde{\mathfrak{B}}) \geq \arcsin(\widetilde{\gamma})$. To prove statement 2, note that the projections $\Pi_{\mathfrak{B}}$ and $\Pi_{\mathfrak{B}'}$ are injective if $\theta(\mathfrak{B},\mathfrak{B}') < \pi/2$. Let $\mathbf{w} \in \mathfrak{B}$. By injectivity of $\Pi_{\mathfrak{B}}$, there exists $\mathbf{w}'' \in \mathfrak{B}'$ such that $\Pi_{\mathfrak{B}}\mathbf{w}'' = \mathbf{w}$. Then $\theta(w,\mathfrak{B}') \leq \theta(\mathbf{w}'',\mathfrak{B}) \leq \sup_{\mathbf{w}'\in\mathfrak{B}'} \theta(\mathbf{w}',\mathfrak{B})$. Taking the supremum over all $\mathbf{w} \in \mathfrak{B}$ thus yields that $\sup_{\mathbf{w}\in\mathfrak{B}} \theta(\mathbf{w},\mathfrak{B}') \leq \sup \mathbf{w}' \in \mathfrak{B}'\theta(\mathbf{w}',\mathfrak{B})$. A similar argument yields the converse inequality, which proves statement 2. Statement 4 follows from the identity $\pi/2 = \sup_{\mathbf{w}\in\mathfrak{B}} \theta(\mathbf{w},\mathfrak{B}') + \inf_{\widetilde{\mathbf{w}}\in\widetilde{\mathfrak{B}},\widetilde{\mathbf{w}}\neq\mathbf{0}} \theta(\widetilde{\mathbf{w}},\mathfrak{B}')$. □

The angle-criterion is a robust criterion in the following sense. If the angle between the system $\mathfrak{B}$ and $\mathfrak{B}'$ is small, then for *every* system trajectory in $\mathfrak{B}$ there exist accurate approximations in $\mathfrak{B}'$. Conversely, *no* trajectory in $\mathfrak{B}'$ is far away from $\mathfrak{B}$.

Example 3.3
We compute the angle between $\mathfrak{D}$ and the static system

$$\mathfrak{C} := \{\mathbf{w} = \left(\begin{smallmatrix}\mathbf{u}\\ \mathbf{y}\end{smallmatrix}\right) \in \ell_2^2 \mid \mathbf{y}_t = \mathbf{u}_t\}.$$

First we compute the orthogonal projection of an element in $\mathfrak{D}$ onto $\mathfrak{C}$. As $\mathfrak{C}$ is a static system, this projection can be carried out pointwise: the projection of $\mathbf{w}_t = (\mathbf{u}_t,\mathbf{y}_t) \in \mathfrak{D}$ is given by $\widehat{\mathbf{w}}_t = (\frac{1}{2}(\mathbf{u}_t + \mathbf{y}_t), \frac{1}{2}(\mathbf{u}_t + \mathbf{y}_t))$. Hence the angle of $(\mathbf{u},\mathbf{y}) \in \mathfrak{D}$ with respect to $\mathfrak{C}$ is the angle between $(\mathbf{u},\mathbf{y})$ and $(\frac{1}{2}(\mathbf{u}+\mathbf{y}), \frac{1}{2}(\mathbf{u}+\mathbf{y}))$. This angle is given by $\arccos(\|\widehat{\mathbf{w}}\|/\|\mathbf{w}\|) = \arccos(\frac{1}{2}\sqrt{2}) = \pi/4$. This angle is not flat.

3.2 Problem formulation

The notions introduced so far lead to the following problem formulation.

DEFINITION 3.4 (OPTIMAL ANGLE REDUCTION (OAR) PROBLEM)
Given a system $\mathfrak{B} \in \mathbb{L}$ with rank m and degree n, and an integer $n' < n$. The optimal angle reduction problem *amounts to determining a system $\mathfrak{R} \in \mathbb{L}$ with the same rank m and degree at most n', such that the angle $\theta(\mathfrak{B}, \mathfrak{R})$ is minimized. Any such system $\mathfrak{R}$ is called an* optimal degree n' approximant *of $\mathfrak{B}$.*

In this paper we will characterize the optimal degree $(n-1)$ angle approximants of systems $\mathfrak{B} \in \mathbb{L}$ of degree n. The following characterizations of optimal approximants are immediate from Proposition 2.7 and Lemma 3.2. Let $(\mathrm{rev}(\mathbf{w}))(t) := \mathbf{w}_{-t}$ be the time reversal operator and let $\mathfrak{B}^{\mathrm{rev}} := \{\mathbf{w} \mid \mathrm{rev}(\mathbf{w}) \in \mathfrak{B}\}$ denote the time reversed system $\mathfrak{B}$.

COROLLARY 3.5
Let $\mathfrak{B} \in \mathbb{L}$. The following conditions are equivalent.

1. *$\mathfrak{R}$ is an optimal approximant of $\mathfrak{B}$.*
2. *$\mathfrak{R}^{\perp}$ is an optimal approximant of $\mathfrak{B}^{\perp}$.*
3. *$\mathfrak{R}^{rev}$ is an optimal aproximant of $\mathfrak{B}^{rev}$.*

4 Cutting links between past and future

4.1 Past-future links

In this section we introduce the system structures that are relevant for the model approximation problem. Let $\mathfrak{B} \in \mathbb{L}$ be a given system and define its past and future behavior as

$$\begin{aligned} \mathfrak{B}^- &:= \mathfrak{B}_{\mathbb{Z}_-} = \mathfrak{B}_{(-\infty,-1]} \\ \mathfrak{B}^+ &:= \mathfrak{B}_{\mathbb{Z}_+} = \mathfrak{B}_{[0,\infty)}. \end{aligned} \tag{4.1}$$

Obviously, for every concatenated trajectory $\mathbf{w}^- \wedge \mathbf{w}^+$ belonging to $\mathfrak{B}$ its past $\mathbf{w}^-$ belongs to $\mathfrak{B}^-$ and its future $\mathbf{w}^+$ to $\mathfrak{B}^+$. The converse, however, is not true: $\mathbf{w}^- \in \mathfrak{B}^-$, $\mathbf{w}^+ \in \mathfrak{B}^+$ does not imply that the concatenation $\mathbf{w}^- \wedge \mathbf{w}^+$ belongs to $\mathfrak{B}$. Indeed, dynamical systems are characterized by the property that their memory structure causes past and future behavior to be linked. In this section we discuss some qualitative and quantitative aspects of the memory structure. The trajectories $\mathbf{w}^- \in \mathfrak{B}^-$ and $\mathbf{w}^+ \in \mathfrak{B}^+$ are said to be *compatible* (or *linked*) if their concatenation $\mathbf{w}^- \wedge \mathbf{w}^+ \in \mathfrak{B}$. For any such compatible pair, $\mathbf{w}^+$ is said to be a *minimal future* of $\mathbf{w}^-$ if its norm, $\|\mathbf{w}^+\|$, is minimal among all compatible futures of $\mathbf{w}^-$. The notion of a *minimal past* is similarly defined.

DEFINITION 4.1 (PAST-FUTURE LINKS)
A past-future link of a system $\mathfrak{B}$ is a system trajectory $\mathbf{w} = \mathbf{w}^- \wedge \mathbf{w}^+ \in \mathfrak{B}$ in which $\mathbf{w}^-$ is a minimal past of $\mathbf{w}^+$ and $\mathbf{w}^+$ a minimal future of $\mathbf{w}^-$. The set of all past-future links of $\mathfrak{B}$ is denoted by $\mathfrak{B}^{\Leftrightarrow}$. The set of all minimal futures of trajectories in $\mathfrak{B}^-$ is denoted by $\mathfrak{B}^{\Rightarrow}$. Similarly, $\mathfrak{B}^{\Leftarrow}$ denotes the set of all minimal pasts.

Note that $\mathfrak{B}^{\Rightarrow} = [\mathfrak{B}^{\Leftrightarrow}]^{+}$ and $\mathfrak{B}^{\Leftarrow} = [\mathfrak{B}^{\Leftrightarrow}]^{-}$. Clearly, a past trajectory may or may not be compatible with a zero future. Similarly, futures (i.e. trajectories in $\mathfrak{B}^{+}$) may or may not be compatible with a zero past. To distinguish between these trajectories we introduce what we will call the *left-* and *right-part* of the system.

$$\begin{aligned} \mathfrak{B}^{\leftarrow\mathbf{0}} &:= \{\mathbf{w} \in \mathfrak{B} \mid \mathbf{w}_t = 0 \text{ for } t \geq 0\} \text{ and} \\ \mathfrak{B}^{\mathbf{0}\rightarrow} &:= \{\mathbf{w} \in \mathfrak{B} \mid \mathbf{w}_t = 0 \text{ for } t < 0\} \end{aligned} \tag{4.2}$$

The idea is that these sets reflect pasts that bring the system into its equilibrium, or futures that can emerge from rest. In the next proposition we summarize some basic properties of past-future links.

PROPOSITION 4.2 (PAST-FUTURE LINKS)
Let $\mathfrak{B} \in \mathbb{L}$ have complexity (m, n). Then

1. $\mathfrak{B} = \mathfrak{B}^{\leftarrow\mathbf{0}} \oplus \mathfrak{B}^{\Leftrightarrow} \oplus \mathfrak{B}^{\mathbf{0}\rightarrow}$.
2. $\dim(\mathfrak{B}^{\Leftrightarrow})$, $\dim(\mathfrak{B}^{\Leftarrow})$ *and* $\dim(\mathfrak{B}^{\Rightarrow})$ *are all finite and equal to* n.
3. *if* $\mathbf{w} \in \mathfrak{B}$ *then* $\mathbf{w} \perp \mathfrak{B}^{\Leftrightarrow}$ *is equivalent to either* $\mathbf{w}^{-} \perp \mathfrak{B}^{\Leftarrow}$ *or* $\mathbf{w}^{+} \perp \mathfrak{B}^{\Rightarrow}$.
4. $\sigma^{j}\mathfrak{B}^{\Rightarrow} \subseteq \mathfrak{B}^{\Rightarrow}$ *and* $\sigma^{-j}\mathfrak{B}^{\Leftarrow} \subseteq \mathfrak{B}^{\Leftarrow}$ *for all* $j \in \mathbb{Z}_{+}$.
5. $\mathbf{w} \in \mathfrak{B}^{\Leftrightarrow}$, *and* $\mathbf{w}_{[0,n]} = \mathbf{0}$ *implies* $\mathbf{w} = \mathbf{0}$.

PROOF. 1. First observe that $\mathfrak{B}^{\leftarrow\mathbf{0}} \perp \mathfrak{B}^{\mathbf{0}\rightarrow}$. Since $\mathfrak{B}^{\Leftrightarrow} = \mathfrak{B} \cap (\mathfrak{B}^{\leftarrow\mathbf{0}} + \mathfrak{B}^{\mathbf{0}\rightarrow})^{\perp}$ it follows that $\mathfrak{B} = \mathfrak{B}^{\leftarrow\mathbf{0}} \oplus \mathfrak{B}^{\Leftrightarrow} \oplus \mathfrak{B}^{\mathbf{0}\rightarrow}$ as desired.

2. Equality of the dimension of the sets follows from the fact that minimal futures are functions of minimal pasts and vice versa. Equality to the degree of $\mathfrak{B}$ follows from Statement 6.

3. For $\mathbf{w} \in \mathfrak{B}$ with $\mathbf{w}^{-} \perp \mathfrak{B}^{\Leftarrow}$, the first statement implies that $(\mathbf{w}^{-} \wedge \mathbf{0}) \in \mathfrak{B}$, hence $(\mathbf{0} \wedge \mathbf{w}^{+}) \in \mathfrak{B}$, hence $\mathbf{w}^{+} \perp \mathfrak{B}^{\Rightarrow}$.

4. Let $\mathbf{p} \in \mathfrak{B}^{-}$ and $\mathbf{f} \in \mathfrak{B}^{+}$ be compatible in $\mathfrak{B}$, i.e., so that $(\mathbf{p} \wedge \mathbf{f}) \in \mathfrak{B}$. If $\mathbf{f}$ is a minimal future of $\mathbf{p}$, then for any $j > 0$ the tail $\mathbf{f}_{[j,\infty)}$ is a minimal norm continuation of $\mathbf{p} \wedge \mathbf{f}_{[0,j-1]}$. By time invariance of $\mathfrak{B}$, the trajectory $[\sigma^{j}(\mathbf{p} \wedge \mathbf{f})]_{(-\infty,-1]}$ belongs to $\mathfrak{B}^{-}$ so that $\sigma^{j}\mathbf{f} \in \mathfrak{B}^{+}$ is its minimal future. Hence, for any $j > 0$, $\sigma^{j}\mathbf{f} \in \mathfrak{B}^{\Rightarrow}$ whenever $\mathbf{f} \in \mathfrak{B}^{\Rightarrow}$. The second inclusion is proven in a similar way.

5. Suppose $\mathbf{w}_{[0,n]} = \mathbf{0}$ and $\mathbf{w}^{+} \neq \mathbf{0}$, then $\{\mathbf{w}^{+}, \ldots, \sigma^{n}\mathbf{w}^{+}\}$ would be a set of $n+1$ independent elements of the n-dimensional space $\mathfrak{B}^{\Rightarrow}$. □

Of particular interest is the relation between past-future links in a system and its orthogonal complement.

PROPOSITION 4.3 (LINKS IN ORTHOGONAL COMPLEMENT)
Let $\mathfrak{B}$ be a system in $\mathbb{L}$ with orthogonal complement $\widetilde{\mathfrak{B}}$. Then

1. $\mathfrak{B}^{\Leftarrow} = \widetilde{\mathfrak{B}}^{\Leftarrow}$, $\mathfrak{B}^{\Rightarrow} = \widetilde{\mathfrak{B}}^{\Rightarrow}$
2. $\mathfrak{B}^{\Leftarrow} = \mathfrak{B}^{-} \cap \widetilde{\mathfrak{B}}^{-}$, $\mathfrak{B}^{\Rightarrow} = \mathfrak{B}^{+} \cap \widetilde{\mathfrak{B}}^{+}$
3. $\mathfrak{B}^{\Leftrightarrow} \perp \widetilde{\mathfrak{B}}^{\Leftrightarrow}$

PROOF.

1. $\mathfrak{B}^{\Leftarrow}$ and $\widetilde{\mathfrak{B}}^{\Leftarrow}$ are both the orthogonal complement of $\mathfrak{B}^{\leftarrow\mathbf{o}} \oplus \widetilde{\mathfrak{B}}^{\leftarrow\mathbf{o}}$ in ℓ_2^-, so they must be identical.
2. In view of the previous, clearly $\mathfrak{B}^{\Leftarrow}$ is contained in $\mathfrak{B}^- \cap \widetilde{\mathfrak{B}}^-$. Equality follows from equality of dimensions.
3. Trivial, as every trajectory in $\mathfrak{B}$ is orthogonal to every trajectory in $\widetilde{\mathfrak{B}}$.

□

Summarizing, three concepts coincide: a minimal past in a system, a minimal past in its orthogonal complement, and the intersection of past behavior of a system and past behavior of its orthogonal complement. Therefore, the only difference between a system and its orthogonal complement is the way in which the minimal pasts are linked with their minimal futures.

4.2 Weakest past-future links

As we shall see, optimal degree-one approximations of a system are determined by the weakest link between past and future in a system. This notion is defined as follows.

DEFINITION 4.4 (WEAKEST GAIN AND WEAKEST LINK)
The weakest forward *and* weakest backward gain *of a system* $\mathfrak{B} \in \mathbb{L}$ *is defined as, respectively,*

$$\overrightarrow{\rho} := \min\{\frac{\|\mathbf{f}\|}{\|\mathbf{p}\|} \mid \mathbf{0} \neq (\mathbf{p} \wedge \mathbf{f}) \in \mathfrak{B}^{\Leftrightarrow}\}$$
$$\overleftarrow{\rho} := \min\{\frac{\|\mathbf{p}\|}{\|\mathbf{f}\|} \mid \mathbf{0} \neq (\mathbf{p} \wedge \mathbf{f}) \in \mathfrak{B}^{\Leftrightarrow}\}$$

Weakest forward and backward links in $\mathfrak{B}$ *are past-future links that achieve the ratios* $\overrightarrow{\rho}$ *and* $\overleftarrow{\rho}$*, respectively. The* weakest gain*,* ρ*, of* $\mathfrak{B}$ *is the minimum of* $\overrightarrow{\rho}$ *and* $\overleftarrow{\rho}$*, and* weakest links *are weakest forward or backward links that achieve this ratio.*

Hence, a weakest forward gain quantifies the minimal relative size of futures versus pasts in the set of all past-future links of the system. Further notice that $0 < \rho \leq 1$, and that $\rho = 1$ implies that all past-future ratios in $\mathfrak{B}^{\Leftrightarrow}$ are one.

EXAMPLE 4.5
A weakest link $\mathbf{p} \wedge \mathbf{f} \in \mathfrak{D}$ is given by

$$\mathbf{p}_t := \begin{bmatrix} 2 \\ -1+\sqrt{5} \end{bmatrix} \left(\frac{1}{2}(3-\sqrt{5})\right)^{-t-1}, \qquad t \in \mathbb{Z}_-$$
$$\mathbf{f}_t := \begin{bmatrix} 3-\sqrt{5} \\ 1-\sqrt{5} \end{bmatrix} \left(\frac{1}{2}(3-\sqrt{5})\right)^{t}, \qquad t \in \mathbb{Z}_+.$$

The time series belongs to $\mathfrak{D}$, its past, $\mathbf{p}$ is orthogonal to $\mathfrak{D}^{\leftarrow\mathbf{o}}$ and hence of minimal size. Similarly, $\mathbf{f}$ is orthogonal to $\mathfrak{D}^{\mathbf{o}\rightarrow}$ and is therefore also of minimal norm. In fact, $\mathbf{p} \wedge \mathbf{f}$ is a weakest forward as well as a weakest backward link, and the set of all those links is given by scalar multiples of this time series. This is always the case for systems of degree one. Here, the weakest backward gain $\overleftarrow{\rho} = \|\mathbf{p}\|/\|\mathbf{f}\| = \sqrt{\frac{1}{2}(3+\sqrt{5})}$ and the weakest forward gain $\overrightarrow{\rho} = 1/\overleftarrow{\rho} = \sqrt{\frac{1}{2}(3-\sqrt{5})}$, which is the smallest and hence equal to the weakest gain ρ of $\mathfrak{D}$.

4.3 Canonical links and ratios

The weakest backward and forward gain of a system $\mathfrak{B} \in \mathbb{L}$ determine the bounds for all past-future ratios $\|\mathbf{p}\|/\|\mathbf{f}\|$ in past-future links: they are in between $\overleftarrow{\rho}$ and $\overrightarrow{\rho}^{-1}$. We refine these notions of extreme ratios to a set of non-decreasing past-future ratios

$$\overleftarrow{\rho} =: \rho_1 \leq \ldots \leq \rho_i \leq \ldots \leq \rho_n = \overrightarrow{\rho}^{-1} \tag{4.3}$$

with n equal to the degree of the system. These numbers are called the *canonical past-future ratios* of $\mathfrak{B}$.

Definition 4.6 (Canonical past-future ratios and links)
The canonical past-future ratios $\rho_1, \ldots, \rho_n$ *and the* canonical past-future links $\widehat{\mathbf{w}}_{(1)}, \ldots, \widehat{\mathbf{w}}_{(n)}$ *of* $\mathfrak{B}$*, are defined recursively by setting* $\rho_1 = \overleftarrow{\rho}$ *and* $\widehat{\mathbf{w}}_{(1)} \in \mathfrak{B}^{\Leftrightarrow}$ *equal to a weakest backward link in* $\mathfrak{B}$ *and*

$$\rho_k := \min\{\frac{\|\mathbf{p}\|}{\|\mathbf{f}\|} \mid 0 \neq (\mathbf{p} \wedge \mathbf{f}) \in \mathfrak{B}^{\Leftrightarrow} \text{ and } (\mathbf{p} \wedge \mathbf{f}) \perp \widehat{\mathbf{w}}_{(i)}, \text{ for } i = 1, \ldots, k-1\},$$

where $\widehat{\mathbf{w}}_{(i)} \in \mathfrak{B}^{\Leftrightarrow}$, $i = 1, \ldots, k-1$ *is such that*

$$\rho_i = \{\frac{\|\widehat{\mathbf{w}}^{-}_{(i)}\|}{\|\widehat{\mathbf{w}}^{+}_{(i)}\|} \text{ and } \widehat{\mathbf{w}}_{(i)} \perp \widehat{\mathbf{w}}_{(j)}, \text{ for } j < i.$$

In addition, we say that a ρ_k *has* multiplicity r *if the number* ρ_k *occurs precisely* r *times in (4.3).*

The following proposition shows that the canonical links of a system and the canonical links of its orthogonal complement are closely related.

Proposition 4.7 (Canonical past-future links)
Let $\mathfrak{B} \in \mathbb{L}$ *have degree* n *and let* $\{\rho_k\}_{k=1}^n$ *and* $\{\tilde{\rho}_k\}_{k=1}^n$ *be the canonical past-future ratios of* $\mathfrak{B}$ *and* $\widetilde{\mathfrak{B}} = \mathfrak{B}^{\perp}$*, respectively. Then*

1. $\rho_k = \tilde{\rho}^{-1}_{n-k+1}$ *for* $k = 1, \ldots, n$.
2. *there exist orthonormal bases* $\mathbf{P} := \{\mathbf{p}_{(1)}, \ldots, \mathbf{p}_{(n)}\}$ *for* $\mathfrak{B}^{\Leftarrow}$ *and* $\mathbf{F} := \{\mathbf{f}_{(1)}, \ldots, \mathbf{f}_{(n)}\}$ *for* $\mathfrak{B}^{\Rightarrow}$ *such that the trajectories*

$$\widehat{\mathbf{w}}_{(k)} := \widetilde{\gamma}_k \mathbf{p}_{(k)} \wedge \gamma_k \mathbf{f}_{(k)}, \tag{4.4}$$
$$\widetilde{\mathbf{w}}_{(k)} := \gamma_k \mathbf{p}_{(k)} \wedge -\widetilde{\gamma}_k \mathbf{f}_{(k)} \tag{4.5}$$

where $k = 1, \ldots, n$ *and*

$$\gamma_k^2 := \frac{1}{1+\rho_k^2}; \quad \text{and} \quad \widetilde{\gamma}_k^2 := \frac{\rho_k^2}{1+\rho_k^2}.$$

define orthonormal bases $\mathbf{X} := \{\widehat{\mathbf{w}}_{(1)}, \ldots, \widehat{\mathbf{w}}_{(n)}\}$ *and* $\widetilde{\mathbf{X}} := \{\widetilde{\mathbf{w}}_{(1)}, \ldots, \widetilde{\mathbf{w}}_{(n)}\}$ *of* $\mathfrak{B}^{\Leftrightarrow}$ *and* $\widetilde{\mathfrak{B}}^{\Leftrightarrow}$*, respectively. Moreover,* $\mathbf{X}$ *and* $\widetilde{\mathbf{X}}$ *are the canonical past-future links corresponding to the canonical ratios* $\{\rho_k\}_{k=1}^n$ *and* $\{\tilde{\rho}_k\}_{k=1}^n$ *and whenever all canonical past-future ratios are distinct, these bases are unique modulo* n *sign changes in one of the four bases.*

PROOF. Let $\phi : \mathfrak{B}^{\Rightarrow} \to \mathfrak{B}^{\Leftarrow}$ be the mapping that associates with $\mathbf{f} \in \mathfrak{B}^{\Rightarrow}$ its minimal compatible antecedent $\mathbf{p} \in \mathfrak{B}^{\Leftarrow}$ in $\mathfrak{B}$. Since both $\mathfrak{B}^{\Rightarrow}$ and $\mathfrak{B}^{\Leftarrow}$ are n-dimensional, ϕ has finite rank and admits a diadic expansion of the form $\phi = \sum_{k=1}^n \sigma_k \mathbf{p}_{(k)} \langle \mathbf{f}_{(k)}, \cdot \rangle$ with $\sigma_1 \geq \ldots \geq \sigma_n$ its singular values. Now take for $\mathbf{P}$ and $\mathbf{F}$ resp. these left- and right singular vectors, which are orthonormal bases for $\mathfrak{B}^{\Leftarrow}$ and $\mathfrak{B}^{\Rightarrow}$. It follows that $\sigma_k = 1/\rho_k$, and that $\mathbf{X}$ is indeed an orthonormal basis of $\mathfrak{B}^{\Leftrightarrow}$ consisting of canonical links. Moreover, σ_n denotes the smallest singular value of ϕ, and hence $\overrightarrow{\rho} = \sigma_n$, which proves the equation in (4.3). Using Proposition 4.3 it is easily deduced that $\widetilde{\mathbf{X}}$ is an orthonormal basis of $\widetilde{\mathfrak{B}}^{\Leftrightarrow}$. Moreover, the quotient

$$\frac{\|\gamma_k \mathbf{p}_{(k)}\|}{\|\widetilde{\gamma}_k \mathbf{f}_{(k)}\|} = \frac{\gamma_k}{\widetilde{\gamma}_k} = \rho_k^{-1}$$

defines a canonical past-future ratio, namely $\tilde{\rho}_{n-k+1}$, of $\widetilde{\mathfrak{B}}$. This proves the result. □

It follows from Proposition 4.7 that canonical past-future links of a system and its orthogonal complement only differ in a scaling factor of their pasts and futures. In particular, the weakest forward and weakest backward gain of a system are equal to the weakest backward and weakest forward gain of its orthogonal complement, and the weakest gain of a system equals the weakest gain of its orthogonal complement.

4.4 Cutting canonical links

In this section we analyze the effect of cutting past-future links of a system. Here, 'cutting' will mean annihilating either the past or the future in a past-future link. Let $\mathfrak{B} \in \mathbb{L}$ have degree n and suppose that $\mathbf{P} = \{\mathbf{p}_{(1)}, \ldots, \mathbf{p}_{(n)}\}$ and $\mathbf{F} = \{\mathbf{f}_{(1)}, \ldots, \mathbf{f}_{(n)}\}$ are orthonormal bases of $\mathfrak{B}^{\Leftarrow}$ and $\mathfrak{B}^{\Rightarrow}$ with properties as stated in Proposition 4.7. Consider, for $k = 1, \ldots, n$ the following systems:

$$\mathfrak{B}\left(\mathbf{p}_{(k)} \wedge \mathbf{0}\right) \tag{4.6}$$

$$\mathfrak{B}\left(\mathbf{0} \wedge \mathbf{f}_{(k)}\right) \tag{4.7}$$

$$\mathfrak{B}\left(\widetilde{\gamma}_k \mathbf{p}_{(k)} \wedge \gamma_k \mathbf{f}_{(k)}\right) = \mathfrak{B}\left(\widehat{\mathbf{w}}_{(k)}\right) \tag{4.8}$$

$$\mathfrak{B}\left(\gamma_k \mathbf{p}_{(k)} \wedge -\widetilde{\gamma}_k \mathbf{f}_{(k)}\right) = \mathfrak{B}\left(\widetilde{\mathbf{w}}_{(k)}\right) \tag{4.9}$$

By construction, $\mathfrak{B}(\widehat{\mathbf{w}}_{(k)}) \subseteq \mathfrak{B}$ and $\mathfrak{B}(\widetilde{\mathbf{w}}_{(k)}) \subseteq \widetilde{\mathfrak{B}}$ and equality holds whenever these systems have rank $m = 1$. The following proposition provides a main tool in the construction of optimal approximate systems.

PROPOSITION 4.8
Let $\mathfrak{B} \in \mathbb{L}$ *have complexity* (n, m) *and let* $1 \leq k \leq n$*. Then*

1. *the systems (4.6), (4.7),(4.8),(4.9) have rank* 1.
2. *the systems (4.6) and (4.7) have degree at most* $n - 1$.
3. *if* $m = 1$ *then* $\mathfrak{B}(\widehat{\mathbf{w}}_{(k)}) = \mathfrak{B}$
4. *if* $m = q - 1$ *then* $\mathfrak{B}(\widetilde{\mathbf{w}}_{(k)}) = \widetilde{\mathfrak{B}}$

5. $\text{shifts}(\mathbf{p}_{(k)} \wedge \mathbf{0}) \perp \text{shifts}(\mathbf{0} \wedge \mathbf{f}_{(k)})$

6. $\theta(\mathfrak{B}(\widehat{\mathbf{w}}_{(k)}), \mathfrak{B}(\mathbf{p}_{(k)} \wedge \mathbf{0})) = \arcsin(\gamma_k) = \theta(\mathfrak{B}(\widetilde{\mathbf{w}}_{(k)}), \mathfrak{B}(\mathbf{0} \wedge \mathbf{f}_{(k)}))$

7. $\theta(\mathfrak{B}(\widehat{\mathbf{w}}_{(k)}), \mathfrak{B}(\mathbf{0} \wedge \mathbf{f}_{(k)})) = \arcsin(\widetilde{\gamma}_k) = \theta(\mathfrak{B}(\widetilde{\mathbf{w}}_{(k)}), \mathfrak{B}(\mathbf{p}_{(k)} \wedge \mathbf{0}))$

Proof. To simplify notation, let $\mathbf{f} = \mathbf{f}_{(k)}$ and $\mathbf{p} = \mathbf{p}_{(k)}$. By Proposition 2.10, the first statement is proven if it is shown that all generating trajectories are of finite degree (cf. (2.4)). To see this, infer from statement 4 in Proposition 4.2 that $\{(\sigma^j(\mathbf{0} \wedge \mathbf{f}))_{[1,\infty)}\}_{j\in\mathbb{N}} \subset \mathfrak{B}^{\Rightarrow}$. Hence, $(\mathbf{0} \wedge \mathbf{f})$ has forward degree at most n (and zero backward degree). Similarly, $(\mathbf{p} \wedge \mathbf{0})$ has backward degree at most n. Finally, $(\alpha\mathbf{p} \wedge \beta\mathbf{f})$ with $\alpha, \beta \in \mathbb{R}$ has back- and forward degree at most n, hence its total degree is at most $2n$. The bound on the degree, in the second statement, is proven as follows. Let $N > n$ and note that

$$\mathfrak{B}(\mathbf{0} \wedge \mathbf{f})_{[0,N]} = \left(\text{span}\{[\sigma^{-j}(\mathbf{0} \wedge \mathbf{f})]_{[0,\infty)}\}_{j=0,\dots,N}\right)_{[0,N]} + \left(\text{span}\{[\sigma^{j}(\mathbf{0} \wedge \mathbf{f})]_{[0,\infty)}\}_{j\in\mathbb{N}}\right)_{[0,N]}.$$

The second set in the right-hand side is contained in $\mathfrak{B}^{\Rightarrow}$. As $(\mathbf{0} \wedge \mathbf{f})_{[0,N]}$ is an element of both sets in the right-hand side, $\dim(\mathfrak{B}(\mathbf{0} \wedge \mathbf{f})_{[0,N]}) \leq (N+1) + (n-1)$, so its degree is at most $n-1$.

Statement 3 and 4 follow from Proposition 2.12.

To prove statement 5, let c_{p_j}, c_{f_j} and c_{pf_j} denote, resp., the correlations $\langle(\mathbf{p} \wedge \mathbf{0}), \sigma^j(\mathbf{p} \wedge \mathbf{0})\rangle$, $\langle(\mathbf{0} \wedge \mathbf{f}), \sigma^j(\mathbf{0} \wedge \mathbf{f})\rangle$, $\langle(\mathbf{p} \wedge \mathbf{0}), \sigma^j(\mathbf{0} \wedge \mathbf{f})\rangle$. Obviously $c_{p_j} = c_{p_{-j}}$, $c_{f_j} = c_{f_{-j}}$ for all $j \in \mathbb{Z}$, and for $j \in \mathbb{N}$, $c_{pf_{-j}} = 0$. Clearly $\text{shifts}(\gamma\mathbf{p} \wedge \widetilde{\gamma}\mathbf{f}) \perp \text{shifts}(\widetilde{\gamma}\mathbf{p} \wedge -\gamma\mathbf{f})$, as these sets belong to $\mathfrak{B}$ and $\widetilde{\mathfrak{B}}$, resp. Orthogonality of the first argument to resp. all left- and right-shifts of the second argument yields, resp., $\gamma\widetilde{\gamma}(c_{p_j} - c_{f_j}) - \gamma^2 c_{pf_j} = 0$ and $\gamma\widetilde{\gamma}(c_{p_j} - c_{f_j}) + \widetilde{\gamma}^2 c_{pf_j} = 0$, for all $j > 0$. Substracting both equations and using $\gamma^2 + \widetilde{\gamma}^2 = 1$ yields $c_{pf_j} = 0$ for all $j > 0$.

Finally we prove the last four equalities. First consider projection of basis elements. We have

$$(\mathbf{p} \wedge \mathbf{0}) = \widetilde{\gamma}(\widetilde{\gamma}\mathbf{p} \wedge \gamma\mathbf{f}) + \widetilde{\gamma}(\gamma\mathbf{p} \wedge -\widetilde{\gamma}\mathbf{f}) \tag{4.10}$$
$$(\mathbf{0} \wedge \mathbf{f}) = \gamma(\widetilde{\gamma}\mathbf{p} \wedge \gamma\mathbf{f}) - \widetilde{\gamma}(\gamma\mathbf{p} \wedge -\widetilde{\gamma}\mathbf{f}) \tag{4.11}$$

which are decompositions of the lefthand side into the projection onto $\mathfrak{B}$ and $\widetilde{\mathfrak{B}}$. This implies that the angle between $(\mathbf{p} \wedge \mathbf{0})$ and $\mathfrak{B}$ is $\arcsin(\gamma)$, and the angle between $(\mathbf{0} \wedge \mathbf{f})$ and $\mathfrak{B}$ is $\arcsin(\widetilde{\gamma})$.

From linearity of the projection it follows that the projection onto $\mathfrak{B}$ of linear combinations

$$\Sigma_{j\in F\subset\mathbb{Z}}\mathbf{v}_j\sigma^j(\mathbf{p} \wedge \mathbf{0}) \in \mathfrak{B}(\mathbf{p} \wedge \mathbf{0})$$

is given by

$$\Sigma_{j\in F\subset\mathbb{Z}}\mathbf{v}_j\sigma^j\gamma(\gamma\mathbf{p} \wedge \widetilde{\gamma}\mathbf{f}) \in \mathfrak{B}.$$

Notice that the relative size of the projection is given by γ, which proves the flatness of the angle for all (finite) linear combinations of shifts of $(\mathbf{p} \wedge \mathbf{0})$. Use (2.5) and the definition of completion to conclude that for every $\widehat{\mathbf{w}} \in \mathfrak{B}$, and $K > 0$ there exist minimal norm extensions $\widehat{\mathbf{w}}^{\leftarrow}$, $\widehat{\mathbf{w}}^{\rightarrow}$ such that $(\widehat{\mathbf{w}}^{\leftarrow} \wedge \widehat{\mathbf{w}}_{[-K,K]} \wedge \widehat{\mathbf{w}}^{\rightarrow})$ belongs to $\mathfrak{B}$ and is also a finite linear combination of these shifts, and hence has the same angle. Now, $\|\widehat{\mathbf{w}}^{\leftarrow}\| \to 0$ and $\|\widehat{\mathbf{w}}^{\rightarrow}\| \to 0$ as $K \to \infty$. Continuity of projection now implies flatness on the whole system. The remaining statements can be proven analogously. □

4.5 Weakest gains with larger multiplicity

The multiplicity of the weakest gain ρ is defined as the multiplicity of the weakest link. If this multiplicity exceeds one, the weakest gain does not determine a unique weakest link. However, the following result shows that the cutted links all generate one and the same system.

PROPOSITION 4.9 (MULTIPLE LINKS)
Let $\mathfrak{B}$ be a system with rank $m = 1$ or $m = q - 1$ and suppose that the weakest gain has multiplicity r. The systems generated by the pasts of all its weakest links are identical, and of degree $n - r$. Similarly, the systems generated by the futures of all its weakest links coincide and are of degree $n - r$.

PROOF. We prove the statement for forward links, and systems of rank 1. For backward links the proof is analogous, and for systems of corank 1 a proof is obtained by interchanging the role of $\mathfrak{B}$ and its orthogonal complement $\widetilde{\mathfrak{B}}$, which then is of rank 1. For $r = 1$ the result is already proved, so we assume $r > 1$.

Let $\mathfrak{L} \subset \mathfrak{B}^{\Leftrightarrow}$ denote the r-dimensional space spanned by the weakest forward links, with $r > 1$.

First we show that there exists an element $\mathbf{w}^* \in \mathfrak{L}$ that belongs to $\sigma^{r-1}\mathfrak{B}^{\Leftrightarrow}$, i.e, for which the continuation after $t = -r$ is minimal. Indeed, the difference between elements in $\mathfrak{L}$ and their minimal continuations starting already at $t = -r + 1$ is of the form

$$\mathbf{0} \underset{-r+1}{\wedge} \bar{\mathbf{w}} \underset{0}{\wedge} \bar{\mathbf{w}}^{+}.$$

This forms an $r - 1$-dimensional space, as $\bar{\mathbf{w}}^{+}$ is the (unique) minimal continuation of its past, and the behavior on finite time intervals of length $r-1$ that can be preceded by zeros is of dimension $m(r-1)$, cf. Proposition 4.2, and, by assumption, $m = 1$. As $\mathfrak{L}$ itself is of higher dimension, it contains an element in $\sigma^{r-1}\mathfrak{B}^{\Leftrightarrow}$. Now observe that this $\mathbf{w}^* \in \mathfrak{L} \cap \sigma^{r-1}\mathfrak{B}^{\Leftrightarrow}$ must have zero values on $[-r+1, 0]$, as otherwise $\sigma^{-r+1}\mathbf{w}^* \in \mathfrak{B}^{\Leftrightarrow}$ would have forward gain below $\overrightarrow{\rho}$. Hence $\{\sigma^{-j}\mathbf{w}^*\}_{j=0,\dots,r-1}$ is a basis for $\mathfrak{L}$, and the pasts of all these basis elements generate the same system $\mathfrak{B}(\mathbf{p}^* \wedge \mathbf{0})$, with $\mathbf{p}^*$ the past of $\mathbf{w}^*$. The degree of the system, $n - r$, is derived from an obvious extension of the proof of Proposition 4.2.2. $\square$

5 Optimal degree-one reductions

The systems (4.6) and (4.7) are candidate systems for approximate models for their degree is strictly smaller than the degree of $\mathfrak{B}$. In fact, the weakest links of $\mathfrak{B}$ define degree $n - 1$ approximants of $\mathfrak{B}$ that turn out to be optimal in the sense of definition 3.4. This is a main result of this paper and stated in the following subsection. Proves are collected in subsection 5.2.

5.1 Solution for systems of rank one

THEOREM 5.1
Let $\mathfrak{B} \in \mathbb{L}^q$ be a system of rank m and degree n, with weakest forward gain $\overrightarrow{\rho}$ and weakest backward gain $\overleftarrow{\rho}$. Suppose that $\overrightarrow{r}$ and $\overleftarrow{r}$ are the multiplicity of $\overrightarrow{\rho}$ and $\overleftarrow{\rho}$ and let $(\overrightarrow{\mathbf{p}} \wedge \overrightarrow{\mathbf{f}})$ and $(\overleftarrow{\mathbf{p}} \wedge \overleftarrow{\mathbf{f}})$ denote a weakest forward and backward link, respectively. Let $\rho = \min(\overrightarrow{\rho}, \overleftarrow{\rho})$ be the weakest gain of $\mathfrak{B}$, and define $\alpha^* := \arctan(\rho)$.

1. *Suppose that $m = 1$ or $m = q-1$ and $\overleftarrow{\rho} \neq \overrightarrow{\rho}$. Then there exists a unique optimal degree $(n-1)$ approximant $\mathfrak{R}^*$ of $\mathfrak{B}$, given by*

$$\mathfrak{R}^* := \begin{cases} \mathfrak{B}(\overrightarrow{\mathbf{p}} \wedge \mathbf{0}) & \text{if } m = 1, \rho = \overrightarrow{\rho} \\ \mathfrak{B}(\mathbf{0} \wedge \overleftarrow{\mathbf{f}}) & \text{if } m = 1, \rho = \overleftarrow{\rho} \\ \mathfrak{B}(\mathbf{0} \wedge \overrightarrow{\mathbf{f}})^\perp & \text{if } m = q-1, \rho = \overrightarrow{\rho} \\ \mathfrak{B}(\overleftarrow{\mathbf{p}} \wedge \mathbf{0})^\perp & \text{if } m = q-1, \rho = \overleftarrow{\rho} \end{cases} \tag{5.1}$$

Moreover, $\theta(\mathfrak{B}, \mathfrak{R}^*) = \alpha^*$, and this angle is flat. $\mathfrak{R}^*$ has degree $n - \overrightarrow{r}$ if $\overrightarrow{\rho} < \overleftarrow{\rho}$, it has degree $n - \overleftarrow{r}$ if $\overrightarrow{\rho} > \overleftarrow{\rho}$.

2. *If $m = 1$ or $m = q-1$ and $\overleftarrow{\rho} = \overrightarrow{\rho}$ then both*

$$\mathfrak{R}_1^* := \begin{cases} \mathfrak{B}(\overrightarrow{\mathbf{p}} \wedge \mathbf{0}) & \text{if } m = 1 \\ \mathfrak{B}(\mathbf{0} \wedge \overrightarrow{\mathbf{f}})^\perp & \text{if } m = q-1 \end{cases}, \quad \text{and} \quad \mathfrak{R}_2^* := \begin{cases} \mathfrak{B}(\mathbf{0} \wedge \overleftarrow{\mathbf{f}}) & \text{if } m = 1 \\ \mathfrak{B}(\overleftarrow{\mathbf{p}} \wedge \mathbf{0})^\perp & \text{if } m = q-1 \end{cases}$$

are optimal degree $(n-1)$ approximants and there are no other solutions. If $\rho = 1$, all optimal approximants are static (i.e., of zero degree).

3. *If $1 \leq m < q$, all optimal degree $(n-1)$ approximants $\mathfrak{R}$ of $\mathfrak{B}$ satisfy*

$$\theta(\mathfrak{R}, \mathfrak{B}) \geq \alpha^*, \tag{5.2}$$

and equality implies that either

$$\overrightarrow{\rho} \leq \overleftarrow{\rho}, (\overrightarrow{\mathbf{p}} \wedge \mathbf{0}) \in \mathfrak{R} \text{ and } (\mathbf{0} \wedge \overrightarrow{\mathbf{f}}) \perp \mathfrak{R} \tag{5.3}$$

or

$$\overleftarrow{\rho} \leq \overrightarrow{\rho}, (\overleftarrow{\mathbf{p}} \wedge \mathbf{0}) \perp \mathfrak{R} \text{ and } (\mathbf{0} \wedge \overrightarrow{\mathbf{f}}) \in \mathfrak{R}. \tag{5.4}$$

The interpretation of this theorem is most straightforward for systems with $m = 1, q = 2$ and $\overrightarrow{\rho} \neq \overleftarrow{\rho}$. Optimal reduction of the state dimension (by at least one degree) amounts to cutting the weakest link of the system, and the unique optimal approximant is generated by the smallest halve (i.e., past or future) of the weakest link. For $q > 2$, and $m = 1$ or $m = q-1$, either the approximant or the error system is of rank one. Optimal degree $(n-1)$ approximants are unique, unless the weakest forward and weakest backward gain coincide. The latter occurs in e.g. time symmetric systems. Systems with weakest gain $\rho = 1$ are 'irreducible', in the sense that the optimal approximant is as good as the error system, both having a flat angle $\pi/4$ with respect to the optimal system. In that case the approximant does not resemble any of the original dynamics, as it has degree 0. For systems with rank $1 < m < q-1$ the result is less specific. We strongly conjecture that the bound α^* is tight.

Theorem 5.1 does not characterize optimal approximants of degree $n' < n-1$. The difficulty in this case is that solutions do no longer have flat angles with respect to the original system. For $n' < n-1$, approximate models can be obtained iteratively by $n - n'$ consecutive approximations in which at each step an optimal approximant of degree one less than the degree of the previous step is obtained. However, such an iterative scheme of sequential reductions will in general not result in an optimal approximate model of degree n'. See Section 6.6 and the example in Section 8.

EXAMPLE 5.2
In Example 4.5 we determined the weakest link in $\mathfrak{D}$. According to Theorem 5.1, the optimal degree 0 approximant of $\mathfrak{D}$ is given by $\mathfrak{R}^* := \mathfrak{B}(\mathbf{p} \wedge \mathbf{0})$ with $\mathbf{p}$ as in Example 4.5. This is a static system given by

$$\mathfrak{R}^* := \{\mathbf{w} = \left(\begin{smallmatrix} \mathbf{u} \\ \mathbf{y} \end{smallmatrix}\right) \in \ell^2 | \left(\begin{smallmatrix} \mathbf{u}_t \\ \mathbf{y}_t \end{smallmatrix}\right) \in \operatorname{im} \left[\begin{smallmatrix} 2 \\ -1+\sqrt{5} \end{smallmatrix}\right]\}.$$

Here, $\theta(\mathfrak{D}, \mathfrak{R}^*) = \arctan(\frac{3-\sqrt{5}}{2})$. The angle is flat, which means that the angle is achieved for every element in $\mathfrak{D}$, and for every element in $\mathfrak{R}^*$. This concludes the leading example.

5.2 Proof

The proof of Theorem 5.1 is structured as follows. First we show how the first part of the theorem, concerning systems of rank $m = 1$ or $m = q - 1$, can be derived as a consequence of statement 3. Then we prove that α^* is indeed a lower bound on achievable angles, by a sequence of lemmas. This involves the construction of trajectories in $\mathfrak{B}$ that define a fixed angle with respect to a given reduced order system $\mathfrak{R}$. From this construction we derive the properties of approximants that achieve the optimal angle α^*.

statement 3 ⇒ statement 1-2.
From Proposition 4.8 it follows that the system defined by (5.1) has degree at most $n - 1$, and a flat angle α^* with respect to $\mathfrak{B}$. By Proposition 4.9, the degree of this system is either $n - \overleftarrow{r}$ or $n - \overrightarrow{r}$. If $\rho = 1$, then all canonical ratios must be one, hence $\rho = \overleftarrow{\rho} = \overrightarrow{\rho} = 1$, and $\overrightarrow{r} = \overleftarrow{r} = n$. In that case, all systems in (5.1) are static and have angle $\pi/4$ with respect to $\mathfrak{B}$. If statement 3 holds, then α^* is optimal and it follows that (5.1) is an optimal degree $(n - 1)$ approximant.

Equations (5.3) and (5.4) specify one time series belonging to a candidate approximant and one belonging to its orthogonal complement. By Proposition 2.12, these time series determine the system completely if $m = 1$ or $m = q - 1$. Consequently, there are no other systems than (5.1) that satisfy (5.3) or (5.4). So it suffices to prove statement 3.

Proof of statement 3.
First we focus on the lower bound (5.2). This lower bound is established by defining trajectories in $\mathfrak{B}$ which achieve this angle with respect to an arbitrary candidate approximate model $\mathfrak{R}$ of degree $n_{\text{red}} < n$. To avoid some technicalities in the proof we assume, throughout this section, that $\theta(\mathfrak{B}, \mathfrak{R}) < \pi/2$. This assumption is justified by the fact that the lower bound on angles never exceeds $\pi/4$, i.e., this assumption valid if there are no reduced order systems with angle below $\pi/2$.

LEMMA 5.3
Let $\mathfrak{B} \in \mathbb{L}^q$ have complexity (m, n) and let $\mathfrak{R} \in \mathbb{L}^q$ have complexity (m, n_{red}) with $n_{red} < n$. Then for every interval $\mathcal{I} \subset \mathbb{Z}$ of length $N \geq n$ there exists $\mathbf{w}^\sharp \in \mathfrak{B}$ of the form

$$\mathbf{w}^\sharp = \mathbf{w}^\leftarrow \wedge \mathbf{w}^* \wedge \mathbf{w}^\rightarrow$$

with the following properties:

1. $\mathbf{0} \neq \mathbf{w}^* \in \mathfrak{B}_\mathcal{I} \cap (\mathfrak{R}_\mathcal{I})^\perp$
2. $\mathbf{w}^\leftarrow$ *and* $\mathbf{w}^\rightarrow$ *are resp. the (unique) minimal backward and forward continuation of* $\mathbf{w}^*$ *in* $\mathfrak{B}$
3. $\|\mathbf{w}^\leftarrow \wedge \mathbf{w}^\rightarrow\| = 1$

PROOF. First we show that $\mathfrak{B}_\mathcal{I} \cap (\mathfrak{R}_\mathcal{I})^\perp$ has positive dimension, which guarantees existence of a non-zero element $\mathbf{w}^*$. By Definition 2.4, $\dim \mathfrak{B}_\mathcal{I} = mN + n$ and $\dim \mathfrak{R}_\mathcal{I} = mN + n_{\text{red}}$. By Proposition 2.7, $(\mathfrak{R}_\mathcal{I})^\perp$ has dimension $qN - (mN + n_{\text{red}})$. Now $\dim \mathfrak{B}_\mathcal{I} + \dim(\mathfrak{R}_\mathcal{I})^\perp = qN + n - n_{\text{red}}$, which implies that there must be an $n - n_{\text{red}}$-dimensional intersection of both. By definition, $\mathfrak{B}_\mathcal{I}$, $\mathbf{w}^*$ admits an extension $\mathbf{w}^{--} \underset{t_0}{\wedge} \mathbf{w}^* \underset{t_1+1}{\wedge} \mathbf{w}^{++} \in \mathfrak{B}$, with $t_0 = \min \mathcal{I}$ and $t_1 = \max \mathcal{I}$. Obviously we may choose $\sigma^{-t_0}\mathbf{w}^{--} \in \mathfrak{B}^{\Leftarrow}$ and $\sigma^{t_1-1}\mathbf{w}^{++} \in \mathfrak{B}^{\Rightarrow}$. As $t_1 - t_0 = N \geq n$, Proposition 4.2 implies that such an extension is unique, so $\sigma^{-t_0}\mathbf{w}^{--} = \mathbf{w}^{\leftarrow}$ and $\sigma^{t_1-1}\mathbf{w}^{++} = \mathbf{w}^{\rightarrow}$. (Uniqueness is not essential here, but it justifies to speak about *the* minimal continuation). Finally, the normalization constraint is justified as, by assumption, $\theta(\mathfrak{B}, \mathfrak{R}) < \pi/2$, and hence $\|\mathbf{w}^{\leftarrow} \wedge \mathbf{w}^{\rightarrow}\| \neq \mathbf{0}$. □

In this way the system can frustrate any approximation on *finite* intervals of any length. This can be translated to the following lower bound on angles.

LEMMA 5.4
Under the hypotheses of Lemma 5.3, $\theta(\mathfrak{B}, \mathfrak{R}) \geq \arctan(\|\mathbf{w}^*\|)$ *with* $\mathbf{w}^*$ *defined in Lemma 5.3.*

PROOF. Since $\mathbf{w}^{\sharp}$, defined in Lemma 5.3, belongs to $\mathfrak{B}$, $\theta(\mathfrak{B}, \mathfrak{R}) \geq \theta(\mathbf{w}^{\sharp}, \mathfrak{R})$. Now let $\widehat{\mathbf{w}}'$ be the orthogonal projection of $\mathbf{w}^{\sharp}$ onto $\mathfrak{R}$, and let $\widetilde{\mathbf{w}}' := \mathbf{w}^{\sharp} - \widehat{\mathbf{w}}'$ be the corresponding error. Then $\theta(\mathbf{w}^{\sharp}, \mathfrak{R}) = \theta(\mathbf{w}^{\sharp}, \widehat{\mathbf{w}}') = \arctan(\frac{\|\widetilde{\mathbf{w}}'\|}{\|\widehat{\mathbf{w}}'\|})$. It remains to prove that $\|\widetilde{\mathbf{w}}'\| \geq \|\mathbf{w}^*\|$, as then

$$\frac{\|\widetilde{\mathbf{w}}'\|^2}{\|\widehat{\mathbf{w}}'\|^2} = \frac{\|\widetilde{\mathbf{w}}'\|^2}{\|\mathbf{w}^{\sharp}\|^2 - \|\widetilde{\mathbf{w}}'\|^2} = \frac{\|\widetilde{\mathbf{w}}'\|^2}{1 + \|\mathbf{w}^*\|^2 - \|\widetilde{\mathbf{w}}'\|^2} \geq \|\mathbf{w}^*\|^2.$$

Therefore we split $\mathbf{w}^{\sharp} = \mathbf{w}^{\text{ext}} + (\mathbf{0} \wedge \mathbf{w}^* \wedge \mathbf{0})$, with $\mathbf{w}^{\text{ext}} = \mathbf{w}^{\leftarrow} \wedge \mathbf{0} \wedge \mathbf{w}^{\rightarrow}$. Let $\widehat{\mathbf{w}}^{ext}$ and $\widetilde{\mathbf{w}}^{ext}$ denote the orthogonal projection of $\mathbf{w}^{\text{ext}}$ onto resp. $\mathfrak{R}$ and $\mathfrak{R}^{\perp}$, so that $\mathbf{w}^{ext} = \widehat{\mathbf{w}}^{ext} + \widetilde{\mathbf{w}}^{ext}$. By construction, $(\mathbf{0} \wedge \mathbf{w}^* \wedge \mathbf{0}) \perp \mathfrak{R}$, so $\widehat{\mathbf{w}}' = \widehat{\mathbf{w}}^{\text{ext}}$ and $\widetilde{\mathbf{w}}' = (\mathbf{0} \wedge \mathbf{w}^* \wedge \mathbf{0}) + \widetilde{\mathbf{w}}^{\text{ext}}$. Further, $\widetilde{\mathbf{w}}^{\text{ext}}_{\mathcal{I}} = -\widehat{\mathbf{w}}^{\text{ext}}_{\mathcal{I}} \in \mathfrak{R}_{\mathcal{I}}$, which implies that $\widetilde{\mathbf{w}}^{\text{ext}} \perp \mathbf{w}^*$. Conclude that $\|\widetilde{\mathbf{w}}'\|^2 = \|\widetilde{\mathbf{w}}^{\text{ext}}\|^2 + \|\mathbf{w}^*\|^2 \geq \|\mathbf{w}^*\|^2$. □

Next we derive a lower bound on the size of $\mathbf{w}^*$ in terms of the weakest gain of the system $\mathfrak{B}$. This bound is independent of $\mathfrak{R}$.

LEMMA 5.5
For $K \in \mathbb{N}$, *let* ρ_K *be the minimum of* $\|\mathbf{w}^{middle}\|$ *over all* $\mathbf{w}^{middle} \in \mathfrak{B}_{[-K,K-1]}$ *for which* $(\mathbf{w}^{\leftarrow} \wedge \mathbf{w}^{middle} \wedge \mathbf{w}^{\rightarrow}) \in \mathfrak{B}$ *with* $\mathbf{w}^{\leftarrow}$ *and* $\mathbf{w}^{\rightarrow}$ *the minimal norm past and future extensions of* $\mathbf{w}^{middle}$ *satisfy* $\|\mathbf{w}^{\leftarrow} \wedge \mathbf{w}^{\rightarrow}\| = 1$. *Then* $\sup_{K \in \mathbb{N}} \rho_K \geq \min(\overrightarrow{\rho}, \overleftarrow{\rho})$.

PROOF. Note that $\sigma^K \mathbf{w}^{\leftarrow} \in \mathfrak{B}^{\Leftarrow}$, and $\sigma^{1-K}\mathbf{w}^{\rightarrow} \in \mathfrak{B}^{\Rightarrow}$. By definition of the weakest forward link, the minimal norm continuation of $\sigma^K \mathbf{w}^{\leftarrow}$ has norm at least $\overrightarrow{\rho}\,\|\mathbf{w}^{\leftarrow}\|$. Similarly, the minimal norm future $\sigma^{1-K}\mathbf{w}^{\rightarrow}$ is only compatible with pasts of norm at least $\overleftarrow{\rho}\,\|\mathbf{w}^{\rightarrow}\|$. The final step in the proof is a limiting argument. As past and future links form finite dimensional spaces of square summable sequences, for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $\mathbf{p} \in \mathfrak{B}^{\Leftarrow}$ and $\mathbf{f} \in \mathfrak{B}^{\Rightarrow}$ $\|\mathbf{p}_{[-N,-1]}\| \geq (1-\epsilon)\|\mathbf{p}\|$ and $\|\mathbf{f}_{[0,N]}\| \geq (1-\epsilon)\|\mathbf{f}\|$. Hence for all $\epsilon > 0$ there also is a K such that $\rho_K \geq (1-\epsilon)\min(\overrightarrow{\rho}, \overleftarrow{\rho})$, from which the bound on the supremum follows. □

The inequality (5.2) can now be proven as follows. Let $\mathfrak{B}$ and $\mathfrak{R}$ satisfy the hypotheses of Lemma 5.3, let $K \geq n$ and let

$$\mathbf{w}^{\sharp}_{(K)} = \mathbf{w}^{\leftarrow} \underset{-K}{\wedge} \mathbf{w}^*_{(K)} \underset{K-1}{\wedge} \mathbf{w}^{\rightarrow} \tag{5.5}$$

be defined as in Lemma 5.3 with $\mathcal{I} = [-K, K-1]$. As $\mathbf{w}^{\sharp}_{(K)} \in \mathfrak{B}$, $\theta(\mathfrak{B}, \mathfrak{R}) \geq \sup_K \theta(\mathbf{w}^{\sharp}_{(K)}, \mathfrak{R})$. From Lemma 5.5 it follows that $\sup_K \|\mathbf{w}^*_{(K)}\| \geq \min(\overleftarrow{\rho}, \overrightarrow{\rho})$. Now Lemma 5.4 implies that $\theta(\mathfrak{B}, \mathfrak{R}) \geq \arctan(\min(\overleftarrow{\rho}, \overrightarrow{\rho})) = \arctan \rho = \alpha^*$.

Finally we need to show that either (5.4) or (5.3) holds for systems that achieve α^*. Suppose $\mathfrak{R}$ satisfies $\theta(\mathfrak{R}, \mathfrak{B}) = \alpha^*$. This implies that for all $K \geq n$, $\theta(\mathbf{w}^{\sharp}_{(K)}, \mathfrak{R}) \leq \alpha^*$, with $\mathbf{w}^{\sharp}_{(K)}$ defined in (5.5). Suppose that $\overrightarrow{\rho} < \overleftarrow{\rho}$. Then this is only possible if $\lim_{K\to\infty} \theta(\sigma^{-K}\mathbf{w}^{\sharp}_{(K)} \wedge \mathbf{0}, \mathfrak{L}) = 0$, where $\mathfrak{L}$ is the f-dimensional space spanned by the pasts of all weakest links. Moreover, since for all $K \geq n$, $(\mathbf{0} \wedge \mathbf{w}^{*}_{(K)} \wedge \mathbf{0}) \perp \mathfrak{R}$, we infer from Lemma 5.4 that also $\lim_{K\to\infty} \theta(\mathbf{w}^{\leftarrow}_{(K)} \wedge \mathbf{0} \wedge \mathbf{w}^{\rightarrow}_{(K)}, \mathfrak{R}) = 0$. Hence, for all $\epsilon > 0$ there is a past $\mathbf{p}$ of a weakest link such that $\theta(\mathbf{p} \wedge \mathbf{0}, \mathfrak{R}) < \epsilon$. As the space spanned by such pasts is of finite dimension (f), and $\mathfrak{R}$ is closed, this implies that $\mathfrak{R}$ must contain a cutted past of some weakest link. Infer from Proposition 4.9 that $\mathfrak{R}$ then contains all separate pasts of these links, as they generate the same system. In a similar way it is derived that $\mathfrak{R}^{\perp}$ must contain the seperate futures of all weakest links.

The proof for $\overleftarrow{\rho}$ is analogous. In case $\overleftarrow{\rho} = \overrightarrow{\rho}$, the same argument implies that still $\mathfrak{R}$ has to contain $(\overrightarrow{\mathbf{p}} \wedge \mathbf{0})$ or $(\mathbf{0} \wedge \overleftarrow{\mathbf{f}})$, and $\mathfrak{R}^{\perp}$ resp. $(\mathbf{0} \wedge \overrightarrow{\mathbf{f}})$ or $(\overleftarrow{\mathbf{p}} \wedge \mathbf{0})$.

6 Isometric state representations

The purpose of this section is to introduce isometric state representations (ISR's) of dynamical systems in the model class $\mathbb{L}$. These representations are further refinements of forward scattering representations (cf. [34]), and have been introduced in [23, 24]. In order to facilitate computational procedures, we provide algorithms for the transformation between classical input/state/output representations and canonical isometric state representations in Section 6.3, and describe how to compute orthogonal projections onto systems in Section 6.4. In Section 6.6 we characterize the process of cutting links in terms of isometric state space representations and provide an algorithm for optimal degree-one reductions for the case where $q = 2$ and $m = 1$.

6.1 Construction and definition

From now on, variables that belong to a system $\mathfrak{B}$ and those belonging to its orthogonal complement $\widetilde{\mathfrak{B}}$ are distinguished by hats and tildes, respectively.

THEOREM 6.1 (CANONICAL ISOMETRIC STATE REPRESENTATIONS)
$\mathfrak{B} \in \mathbb{L}^q$ if and only if there exists $m, n \in \mathbb{N}$ and a square partitioned matrix $M = \begin{bmatrix} A & B & \widetilde{B} \\ C & D & \widetilde{D} \end{bmatrix} \in \mathbb{R}^{(n+m+p)\times(n+q)}$ with $p = q - m$, such that

1. *M is unitary, i.e., $MM^\top = M^\top M = I_{n+q}$.*
2. *$\mathfrak{B} = \{\widehat{\mathbf{w}} \in \ell_2^q \mid \exists \widehat{\mathbf{x}} \in \ell_2^n, \widehat{\mathbf{v}} \in \ell_2^m$ such that*

$$\begin{aligned} \widehat{\mathbf{x}}_{t+1} &= A\widehat{\mathbf{x}}_t + B\widehat{\mathbf{v}}_t \\ \widehat{\mathbf{w}}_t &= C\widehat{\mathbf{x}}_t + D\widehat{\mathbf{v}}_t\}. \end{aligned} \tag{6.1}$$

3. *$\widetilde{\mathfrak{B}} = \{\widetilde{\mathbf{w}} \in \ell_2^q \mid \exists \widetilde{\mathbf{x}} \in \ell_2^n, \widetilde{\mathbf{v}} \in \ell_2^p$ such that*

$$\begin{aligned} \widetilde{\mathbf{x}}_{t+1} &= A\widetilde{\mathbf{x}}_t + \widetilde{B}\widetilde{\mathbf{v}}_t \\ \widetilde{\mathbf{w}}_t &= C\widetilde{\mathbf{x}}_t + \widetilde{D}\widetilde{\mathbf{v}}_t\}. \end{aligned} \tag{6.2}$$

4. *the gramian* $W := \Sigma_{j\in\mathbb{N}} A^j BB^\top A^{\top j}$ *is diagonal with non-increasing diagonal elements* $1 > \lambda_1 \geq \ldots \geq \lambda_n > 0$, *or, equivalently, the gramian* $\widetilde{W} := \Sigma_{j\in\mathbb{N}} A^j \widetilde{B}\widetilde{B}^\top A^{\top j} = I_n - W$ *is diagonal with non-decreasing diagonal elements* $0 < 1-\lambda_n \leq \ldots \leq 1-\lambda_n < 1$.

Proof. This has been proven in [14, 24]. In the context of this paper we provide an independent proof of the 'only if' part as follows. Let $\mathbf{F} := \{\mathbf{0} \wedge \mathbf{f}_{(k)}\}_{k=1}^n$ with $\mathbf{f}_{(k)} \in \mathfrak{B}^{\Rightarrow}$ as defined in Proposition 4.7. Let $\mathbf{V}$ denote an orthonormal basis of $\widehat{\mathfrak{V}} := (\mathfrak{B}^{\Leftrightarrow} + \sigma^{-1}\mathfrak{B}^{\Leftrightarrow}) \cap (\mathfrak{B}^{\Leftrightarrow})^\perp$ and let $\widetilde{\mathbf{V}}$ denote an orthonormal basis of $\widetilde{\mathfrak{V}} := (\widetilde{\mathfrak{B}}^{\Leftrightarrow} + \sigma^{-1}\widetilde{\mathfrak{B}}^{\Leftrightarrow}) \cap (\widetilde{\mathfrak{B}}^{\Leftrightarrow})^\perp$. Define $\mathbf{W}_0 := \{\mathbf{e}_1, \ldots, \mathbf{e}_q\}$, where $\mathbf{e}_k$ has been introduced in the notation section. Then $\mathbf{W}_0$ is an orthonormal basis for the values of time series in ℓ_2^q at $t = 0$. Let M be such that

$$M : \mathbf{T}_1 := \{\mathbf{F}, \mathbf{V}, \widetilde{\mathbf{V}}\} \mapsto \mathbf{T}_2 := \{\sigma^{-1}\mathbf{F}, \mathbf{W}_0\}.$$

We claim that $\mathbf{T}_1$ and $\mathbf{T}_2$ are both orthonormal bases of

$$\mathfrak{H} := \{\mathbf{w} \in \ell_2^q \mid \mathbf{w} = (\mathbf{0} \wedge \mathbf{w}_0 \wedge \mathbf{w}^{\rightarrow}) \text{ with } \mathbf{w}_0 \in \mathbb{R}^q \text{ and } \sigma\mathbf{w}^{\rightarrow} \in \mathfrak{B}^{\Rightarrow}\} \tag{6.3}$$

For $\mathbf{T}_2$ this is obvious. To see this for $\mathbf{T}_1$, we derive that

$$\widehat{\mathfrak{V}} := \{\widehat{\mathbf{w}} \in \mathfrak{B} | \widehat{\mathbf{w}} = (\mathbf{0} \wedge \widehat{\mathbf{w}}_0 \wedge \widehat{\mathbf{w}}^{\rightarrow}) \text{ with } \sigma\widehat{\mathbf{w}}^{\rightarrow} \in \mathfrak{B}^{\Rightarrow}\} \tag{6.4}$$
$$\widetilde{\mathfrak{V}} := \{\widetilde{\mathbf{w}} \in \widetilde{\mathfrak{B}} | \widetilde{\mathbf{w}} = (\mathbf{0} \wedge \widetilde{\mathbf{w}}_0 \wedge \widetilde{\mathbf{w}}^{\rightarrow}) \text{ with } \sigma\widetilde{\mathbf{w}}^{\rightarrow} \in \widetilde{\mathfrak{B}}^{\Rightarrow}\} \tag{6.5}$$

Then $\widehat{\mathfrak{V}} \subset \mathfrak{B}^{\mathbf{o}\rightarrow}$ and $\widetilde{\mathfrak{V}} \subset \widetilde{\mathfrak{B}}^{\mathbf{o}\rightarrow}$ imply orthonormality of $\mathbf{T}_1$. Further, as $\widehat{\mathbf{w}}^{\rightarrow}$ and $\widetilde{\mathbf{w}}^{\rightarrow}$ denote (unique) minimum norm continuations in resp. $\mathfrak{B}$ and $\widetilde{\mathfrak{B}}$, $\dim \widehat{\mathfrak{V}} = \dim \widehat{\mathfrak{V}}_0 = m$ and $\dim \widetilde{\mathfrak{V}} = p = q - m$, cf. Lemma 2.5. So $\mathbf{T}_1$ contains $n + q$ elements, and all obviously belong to $\mathfrak{H}$. Therefore it suffices to prove the equation (6.4) (the proof of (6.5) is analogous).

First observe that $\widehat{\mathfrak{V}} \perp \mathfrak{B}^{\Leftrightarrow}$ implies $\widehat{\mathfrak{V}}^- \perp \mathfrak{B}^{\Leftarrow}$ (the implication is valid for any subset of $\mathfrak{B}$, cf. Proposition 4.2). On the other hand, from the same proposition it follows that $\sigma^{-1}\mathfrak{B}^{\Leftarrow} \subset \mathfrak{B}^{\Leftarrow}$, and hence $\widehat{\mathfrak{V}}^- \subset \mathfrak{B}^{\Leftarrow}$. Conlude that $\widehat{\mathfrak{V}}^- = \mathbf{0}$. Also $(\sigma\widehat{\mathfrak{V}})^+ \subset (\sigma\mathfrak{B}^{\Rightarrow} + \mathfrak{B}^{\Rightarrow}) = \mathfrak{B}^{\Rightarrow}$, which implies that for any $\widehat{\mathbf{w}} \in \widehat{\mathfrak{V}}$, $\widehat{\mathbf{w}}_{[1,\infty)}$ is indeed the (unique) minimal continuation of its past. The equation for $\widetilde{\mathfrak{V}}$ is proved analogously. By consequence, M is a unitary matrix.

The equations (6.1) are valid with $(\widehat{\mathbf{x}}_t, \widehat{\mathbf{v}}_t)$ the coefficients of the projection of $\sigma^j\widehat{\mathbf{w}}$ onto $\mathfrak{H}$ for basis $\mathbf{T}_1$. Namely, for $\widehat{\mathbf{w}} \in \mathfrak{B}$ we have $\widetilde{\mathbf{v}}_t = 0$ for all $t \in \mathbb{Z}$, and by definition of M, $(\widehat{\mathbf{x}}_{t+1}, \widehat{\mathbf{w}}_t)$ are its coefficients with respect to $\mathbf{T}_2$. The proof of (6.2) is analogous.

Finally, the result concerning W and $\widetilde{W}$ is an immediate consequence of the definition of $\mathbf{F}$ in Proposition 4.7. The diagonal elements of the gramian W coincide with the values of γ_k^2 in that proposition. □

The matrix M is a *canonical isometric state representation* (CISR) of $\mathfrak{B}$ (and of $\widetilde{\mathfrak{B}}$). These representations are minimal, in the sense that neither the state dimension nor the number of auxiliary inputs can be reduced. The term 'canonical' is usually related to uniqeuness of representations. If all canonical gains of a system are distinct, the only non-uniqueness in a CISR is a sign transformation of each state component, and a unitary basis transformation of the auxiliary inputs $\mathbf{v}$ and $\widetilde{\mathbf{v}}$.

If M satisfies the conditions 1-3 of Theorem 6.1 then M is called an *isometric state representation* (ISR) of $\mathfrak{B}$. The quadruple (A, B, C, D) in (6.1) is refered to as a *state representation* (SR) of $\mathfrak{B}$, and we write $\mathfrak{B}(A, B, C, D)$. In Section 6.3 we describe how to transform a SR's to an equivalent CISR.

6.2 From CISR to canonical links

It is straightforward to construct the canonical links from a given CISR. Let $M = \left[\begin{smallmatrix} A & B & \widetilde{B} \\ C & D & \widetilde{D} \end{smallmatrix}\right]$ define a CISR of $\mathfrak{B}$ (and of $\widetilde{\mathfrak{B}}$), with controllability gramians W and $\widetilde{W} = I_n - W$ and let $F := AWC^\top + BD^\top$. The k-th canonical link $\widehat{\mathbf{w}}_{(k)} = (\widetilde{\gamma}_k \mathbf{p}_{(k)} \wedge \gamma_k \mathbf{f}_{(k)}) \in \mathfrak{B}$, satisfies the state equations (6.1) with auxiliary input $\widehat{\mathbf{v}}_{(k)}$ and state sequence $\widehat{\mathbf{x}}_{(k)}$ given by

$$\widehat{\mathbf{w}}_{(k)} := \begin{cases} F^\top A^{\top^{-t-1}} W^{-\frac{1}{2}} e_k & \text{if } t \in \mathbb{Z}_- \\ CA^t W^{\frac{1}{2}} e_k & \text{if } t \in \mathbb{Z}_+ \end{cases}; \qquad \widehat{\mathbf{v}}_k := \begin{cases} B^\top A^{\top^{-t-1}} W^{-\frac{1}{2}} e_k & \text{if } t \in \mathbb{Z}_- \\ 0 & \text{if } t \in \mathbb{Z}_+ \end{cases} \tag{6.6}$$

$$\widehat{\mathbf{x}}_k := \begin{cases} WA^{\top^{-t}} W^{-\frac{1}{2}} e_k & \text{if } t \in \mathbb{Z}_- \\ A^t W^{\frac{1}{2}} e_k & \text{if } t \in \mathbb{Z}_+ \end{cases} \tag{6.7}$$

$$\mathbf{p}_{(k)} := F^\top A^{\top^{-t-1}} (W\widetilde{W})^{-\frac{1}{2}} e_k,\ t \in \mathbb{Z}_-; \qquad \mathbf{f}_{(k)} := CA^t e_k,\ t \in \mathbb{Z}_+ \tag{6.8}$$

6.3 Obtaining a CISR

In addition to this abstract construction of CISR's from behaviors, we now describe how to obtain these representations from input/state/output representations (cf. also [23, 24]). Consider the state equations

$$\begin{aligned} \widehat{\mathbf{x}}_{t+1} &= A'\widehat{\mathbf{x}}_t + B'\widehat{\mathbf{u}}_t \\ \widehat{\mathbf{y}}_t &= C'\widehat{\mathbf{x}}_t + D'\widehat{\mathbf{u}}_t, \end{aligned} \tag{6.9}$$

where $\widehat{\mathbf{u}}_t \in \mathbb{R}^m$ and $\widehat{\mathbf{y}}_t \in \mathbb{R}^p$ denote the input and output of the system at time t. Let $\mathfrak{B}_{i/o}(A', B', C', D')$ denote the set of square summable trajectories $(\widehat{u}, \widehat{y})$ that satisfy (6.9). The representation is called minimal if the state dimension cannot be reduced without affecting the system. The next algorithm converts $\mathfrak{B}_{i/o}(A', B', C', D')$ to a CISR.

Data: (A', B', C', D') which defines the system $\mathfrak{B} = \mathfrak{B}_{i/o}(A', B', C', D')$, with m inputs and $p = q - m$ outputs.

Step 1: Construction of an SR
Define $A := A'$, $B := B'$, $C := \begin{bmatrix} 0 \\ C' \end{bmatrix}$, $D := \begin{bmatrix} I_m \\ D' \end{bmatrix}$.

Step 2: Construction of an ISR
Determine matrices F and nonsingular matrices S and R such that

$$\widehat{M} := \begin{bmatrix} S & 0 \\ 0 & I_q \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} S^{-1} & 0 \\ FS^{-1} & R \end{bmatrix} \tag{6.10}$$

is isometric. This is obtained by solving $S^\top S = K$, $RR^\top = P^{-1}$ and taking $F := -P^{-1}L$, with $P := B^\top KB + D^\top D$, $L := B^\top KA + D^\top C$ and K the unique symmetric positive definite solution of the algebraic Riccati equation $K = A^\top KA - L^\top P^{-1}L + C^\top C$. An ISR M is obtained by completing $\widehat{M}$ to a square unitary matrix, by adding the last $q - m$ columns of U with $U\Sigma V^\top$ the SVD of $\widehat{M}$. (Re)define $A, B, C, D, \widetilde{B}, \widetilde{D}$ corresponding to this M.

Step 3: Diagonalize gramian
Determine the unique symmetric solution of $W = AWA^\top + BB^\top$, and its SVD $W = U\Lambda U^\top$, with $W =$

$\mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ and $1 > \lambda_1 > \ldots > \lambda_n > 0$. Redefine M as $\begin{bmatrix} U^\top & 0 \\ 0 & I_q \end{bmatrix} M \begin{bmatrix} U & 0 \\ 0 & I_q \end{bmatrix}$, and redefine $A, B, C, \widetilde{B}$ correspondingly.

Result:
M is a CISR of $\mathfrak{B}$.

Conversely, all SR's (in particular CISR's) can be transformed to i/s/o representations as follows.

Data:
A minimal SR (A, B, C, D) of a system $\mathfrak{B}$.

Step 1:
Rearrange the components such that $D = \begin{bmatrix} D_u \\ D_y \end{bmatrix}$ and $C = \begin{bmatrix} C_u \\ C_y \end{bmatrix}$ with D_u (square and) nonsingular.

Step 2:
Define $A' := A - BD_u^{-1}C_u$, $B' := BD_u^{-1}$, $C' := C_y - D_y D_u^{-1}C_u$, and $D' := D_y D_u^{-1}$.

Result:
(A', B', C', D') is a minimal i/s/o-representation of $\mathfrak{B}$.

See [23] for proofs of these claims. We remark that the first algorithm constructs a canonical isometric state space realization of a normalized coprime factorization of the transfer function associated with the state space system (6.9) (See [30] and [25, Proposition3]). Further, Step 1 in the last algorithm shows that an input-output decomposition of variables is not unique in general.

6.4 Orthogonal projection formula

The orthogonal projection of a time series $\mathbf{w} \in \ell_2^q$ onto a system $\mathfrak{B}$ can be calculated in terms of CISR's as follows.

Proposition 6.2 (Orthogonal Projection)
Let $\mathfrak{B} \in \mathbb{L}$. Every $\mathbf{w} \in \ell_2$ admits a decomposition $\mathbf{w} = \widehat{\mathbf{w}} + \widetilde{\mathbf{w}}$ with $\widehat{\mathbf{w}}$ the orthogonal projection of $\mathbf{w}$ on $\mathfrak{B}$ and $\widetilde{\mathbf{w}}$ the orthogonal projection of $\mathbf{w}$ on $\widetilde{\mathfrak{B}}$. $\widehat{\mathbf{w}}$ and $\widetilde{\mathbf{w}}$ are uniquely determined by (6.1) and (6.2) where $\widehat{\mathbf{v}}$ and $\widetilde{\mathbf{v}}$ are given by

$$\begin{aligned} \mathbf{x}_t &= A^\top \mathbf{x}_{t+1} + C^\top \mathbf{w}_t \\ \widehat{\mathbf{v}} &= B^\top \mathbf{x}_{t+1} + D^\top \mathbf{w}_t \\ \widetilde{\mathbf{v}} &= \widetilde{B}^\top \mathbf{x}_{t+1} + \widetilde{D}^\top \mathbf{w}_t . \end{aligned} \tag{6.11}$$

Proof. Premultiplying (6.11) with M yields

$$\begin{aligned} \mathbf{x}_{t+1} &= A\mathbf{x}_t + B\widehat{\mathbf{v}}_t + \widetilde{B}\widetilde{\mathbf{v}}_t \\ \mathbf{w}_t &= C\mathbf{x}_t + D\widehat{\mathbf{v}}_t + \widetilde{D}\widetilde{\mathbf{v}}_t . \end{aligned} \tag{6.12}$$

Hence $\mathbf{x} = \widehat{\mathbf{x}} + \widetilde{\mathbf{x}}$ and $\mathbf{w} = \widehat{\mathbf{w}} + \widetilde{\mathbf{w}}$ with $\widehat{\mathbf{w}} \in \mathfrak{B}$ and $\widetilde{\mathbf{w}} \in \widetilde{\mathfrak{B}}$, from which the result follows. □

6.5 Computing the angle between systems

In this section we explain how to determine the angle between two systems, cf. Definition 3.1. First observe that for systems with a *flat* angle, which is the most relevant case in this paper, this amounts to the orthogonal projection of an *arbitrary* non-zero trajectory in the one system onto the other one, as described above, and then determine the corresponding angle, cf. Section 3.1.

For the general case we relate the angle between two systems to the H_∞-norm of a transfer function. We remark, however, that the algorithm is not needed in the model reduction approach in this paper.

Data: A CISR of a system $\mathfrak{B}$ and $\mathfrak{B}'$, both of the same rank m, given by resp.

$$\begin{bmatrix} A & B & \widetilde{B} \\ C & D & \widetilde{D} \end{bmatrix} \text{ and } \begin{bmatrix} A' & B' & \widetilde{B}' \\ C' & D' & \widetilde{D}' \end{bmatrix} \tag{6.13}$$

Step 1: Construct the rational transfer functions $G_1(s) := C(sI_n - A)^{-1}B + D$ and $\widetilde{G}_2(s) := C'(I_s - A')^{-1}\widetilde{B}' + \widetilde{D}'$,

Step 2: Determine the $m \times p$ series connection $H := \widetilde{G}_2^* G_1$, with $\widetilde{G}_2^*(s) = \widetilde{G}^\top(s^{-1})$ the adjoint of $\widetilde{G}_2$.

Step 3: Determine the induced norm $\nu := \|H\|_\infty := \sup_{\widehat{\mathbf{v}} \in \ell_2^m} \frac{\|H\widehat{\mathbf{v}}\|}{\|\widehat{\mathbf{v}}\|}$, which is equal to the square root of $\sup_{0 \leq \theta \leq 2\pi} \|H^\top(e^{-i\theta})H(e^{i\theta})\|_\infty$, with the matrix norm defined as the largest eigenvalue of the symmetric matrix.

Result: $\alpha := \arcsin \nu$ is the angle beteen $\mathfrak{B}$ and $\mathfrak{B}'$.

The proof of correctness relies on the fact that H is the mapping from the auxiliary input $\widehat{\mathbf{v}}$ of a system trajectory in $\mathfrak{B}$ to the auxiliary input $\widetilde{\mathbf{v}}'$ of the projection error $\widehat{\mathbf{w}} - \widehat{\mathbf{w}}'$, with $\widehat{\mathbf{w}}'$ the projection of $\widehat{\mathbf{w}}$ onto $\mathfrak{B}'$. As $\widetilde{G}_2$ is isometric, $\|\widetilde{\mathbf{v}}'\|$ is the size of the projection error, from which the result follows.

6.6 Cutting links in state representations

In this section, the effect of cutting a canonical link is translated in terms of state representations. In view of the main result, the systems $\mathfrak{B}(\mathbf{p}_{(k)} \wedge \mathbf{0})$ and $\mathfrak{B}(\mathbf{0} \wedge \mathbf{f}_{(k)})$ are of main interest. These systems have rank 1, and their orthogonal complements have rank $q-1$. In the analysis, we focus attention to $\mathfrak{B}(\mathbf{0} \wedge \mathbf{f}_{(k)})$. Similar results can be inferred for $\mathfrak{B}(\mathbf{p}_{(k)} \wedge \mathbf{0})$ by a time-reversion argument.

In terms of a CISR $M = \begin{bmatrix} A & B & \widetilde{B} \\ C & D & \widetilde{D} \end{bmatrix}$, the trajectory $\mathbf{0} \wedge \mathbf{f}_{(k)}$ is given by

$$\{\ldots, 0, 0 \mid Ce_k, CAe_k, CA^2e_k, \ldots\}.$$

A state representation for $\mathfrak{B}(\mathbf{0} \wedge \mathbf{f}_{(k)})$ is given by

$$\begin{aligned} \sigma\mathbf{z} &= A\mathbf{z} + Ae_k\mathbf{h} \\ \mathbf{w} &= C\mathbf{z} + Ce_k\mathbf{h}. \end{aligned}$$

with $\mathbf{h}$ an auxiliary input that has $\mathbf{f}_{(k)}$ as its impulse response. Notice that this representation is not minimal, as the degree of the system is strictly smaller than the degree n of $\mathfrak{B}$. A reduced order representation is obtained with the auxiliary input $\mathbf{h}' := \mathbf{h} + e_k^\top \mathbf{z}$, leading to

$$\begin{aligned}\sigma \mathbf{z}' &= E_k^\top A E_k \mathbf{z}' + A e_k \mathbf{h}' \\ \mathbf{w} &= C E_k \mathbf{z}' + C e_k \mathbf{h}',\end{aligned}$$

where E_k has been introduced in the notation section.

If the k-th canonical gain has multiplicity one, this is indeed a minimal state representation for $\mathfrak{B}(\mathbf{0} \wedge \mathbf{f}_{(k)})$. Notice, however, that it is not isometric in general.

To obtain a state representation for $\mathfrak{B}(\mathbf{0} \wedge \mathbf{f}_{(k)})^\perp$, we first recall the state equations of $\mathfrak{B}$ induced by the CISR M,

$$\begin{pmatrix} \sigma \mathbf{x} \\ \mathbf{w} \end{pmatrix} = M \begin{pmatrix} \mathbf{x} \\ \widehat{\mathbf{v}} \\ \widetilde{\mathbf{v}} \end{pmatrix} \text{ and } \begin{pmatrix} \mathbf{x} \\ \widehat{\mathbf{v}} \\ \widetilde{\mathbf{v}} \end{pmatrix} = M^\top \begin{pmatrix} \sigma \mathbf{x} \\ \mathbf{w} \end{pmatrix}$$

which are given by (6.12) and (6.11). The condition $\sigma^j \mathbf{w} \perp (\mathbf{0} \wedge \mathbf{f}_{(k)})$ is equivalent to $e_k^\top \mathbf{x}_j = 0$, cf. the proof of Theorem 6.1, and hence

$$\mathbf{w} \in \mathfrak{B}(\mathbf{0} \wedge \mathbf{f}_{(k)})^\perp \text{ iff } e_k^\top \mathbf{x} = \mathbf{0}.$$

This condition can be translated to a state feedback law, by solving

$$e_k^\top (A\mathbf{x} + B\widehat{\mathbf{v}} + \widetilde{B}\widetilde{\mathbf{v}}) = \mathbf{0} \tag{6.14}$$

for one component in the auxiliary inputs in $\widehat{\mathbf{v}}$ or $\widetilde{\mathbf{v}}$. Eliminating this single component then results in a (not necessarily stable) state representation with $q-1$ remaining auxiliary inputs.

We further restrict the attention to systems with $m = q-1$, as then the (single component) $\widetilde{\mathbf{v}}$ can be eliminated completely, by substituting

$$\widetilde{\mathbf{v}} = -\frac{e_k^\top A\mathbf{x}}{e_k^\top \widetilde{B}} - \frac{e_k^\top B\widehat{\mathbf{v}}}{e_k^\top \widetilde{B}}. \tag{6.15}$$

This results in the following state representation for $\mathfrak{B}(\mathbf{0} \wedge \mathbf{f}_{(k)})^\perp$.

Proposition 6.3 (SR of k-th canonical state annihilator)
Suppose that $\mathfrak{B}$ has rank $m = q-1$. Then $\mathfrak{B}(\mathbf{0} \wedge \mathbf{f}_{(k)})^\perp$ has a state representation

$$\begin{aligned}\mathbf{x}_{t+1} &= T_k A \mathbf{x}_t + T_k B \widehat{\mathbf{v}}_t \\ \mathbf{w}_t &= (C - \alpha \widetilde{D} e_n^\top A)\mathbf{x}_t + (D - \alpha \widetilde{D} e_n^\top B)\widehat{\mathbf{v}}_t\end{aligned} \tag{6.16}$$

with $\alpha := 1/(e_k^\top \widetilde{B})$ and $T_k := I_k - \alpha \widetilde{B} e_k^\top$. Its controllability gramian is given by $W - \frac{\lambda_k}{\widetilde{\lambda}_k}\widetilde{W}$. The system is of degree $n-r$, and has at most $k-r'$ stable poles, with r the multiplicity of gain λ_k, and $k-r'$ the index of the first gain strictly smaller than λ_k.

Proof. The formula for the state representation follow directly from the substitution (6.15). The controllability gramian of $(T_k A, T_k B)$ is given by $X = W - \frac{\lambda_k}{\widetilde{\lambda}_k}\widetilde{W}$, as it is a solution of $X = T_k A X A^\top T_k^\top + T_k B B^\top T_k^\top$, which is derived as follows. Observe that $T_k \widetilde{B} = 0$ and $e_k^\top T_k = 0$, and hence $T_k(AXA^\top + BB)T_k^\top =$

$T_k(AXA^\top + BB - \frac{\lambda_k}{\widetilde{\lambda}_k}\widetilde{B}\widetilde{B}^\top)T_k^\top = T_k X T_k^\top = X + (I_n - T_k)X(I_n - T_k)^\top = X$. Finally, notice that the gramian is diagonal, with first $k - r'$ entries positive, then r zero diagonal entries, and the remaining ones negative. This implies that in $[T_k A \ \ T_k B]$ the $k - r' + r$-th to $k - r' + 1$-th row is zero. Removing these r rows yields a minimal representation, as the degree of the represented system is indeed $n - r$, cf. Proposition 4.9, and this has the same controllability gramian with the zero diagonal entries removed. Then infer from Theorem 3.3 in [11] that the rational transfer function which maps $\widehat{\mathbf{v}}$ into $\mathbf{w}$ according to (6.16) has $k - r'$ stable poles. □

We emphasize that the state representation (6.16) is not necessarily isometric. It can be transformed into a CISR as described in Section 6.3. We remark that in general this requires a non-diagonal state space transformation, and that it seems difficult to derive analytic formulas, unless the number of system variables is two.

6.7 Reduced order CISR for siso systems

The following theorem gives an explicit degree-one reduction formula in terms of CISR's for the case where $m = 1$ and $q = 2$. This corresponds to systems with single input and single output, cf. Section 6.3.

THEOREM 6.4 (CISR OPTIMAL REDUCTIONS)
Let $\mathfrak{B} \in \mathbb{L}$ have a CISR $M = \begin{bmatrix} A & B & \widetilde{B} \\ C & D & \widetilde{D} \end{bmatrix}$ and suppose that $q = 2$, rank $m = 1$, $\lambda_n \leq 1 - \lambda_1$, and all λ_j are distinct. Define $\widetilde{\lambda} := \widetilde{\lambda}_n$ and $\lambda := \lambda_n$. A CISR of the optimal angle approximant $\mathfrak{R}^* = \mathfrak{B}(\mathbf{0} \wedge \mathbf{f}_n) = \mathfrak{B}(\mathbf{p}_n \wedge \mathbf{0})$ of $\mathfrak{B}$ is given by

$$M^* := \begin{bmatrix} Q^\sharp TAQ & \widetilde{\lambda}^{\frac{1}{2}} Q^\sharp TB & \beta\widetilde{\lambda}^{\frac{1}{2}} QAe_n \\ (C - \beta\widetilde{D}e_n^\top A)Q & \widetilde{\lambda}^{\frac{1}{2}}(D - \beta\widetilde{D}e_n^\top B) & \beta\widetilde{\lambda}^{\frac{1}{2}} Ce_n \end{bmatrix}_{\bar{n}} =: \begin{bmatrix} A_R & B_R & \widetilde{B}_R \\ C_R & D_R & \widetilde{D}_R \end{bmatrix} \tag{6.17}$$

where $[\cdot]_{\bar{n}}$ denotes the removal of the n-th row and column in a matrix, and where $\beta := 1/(e_n^\top \widetilde{B})$, $T := I_n - \beta\widetilde{B}e_n^\top$, $Q := (I_n - \lambda W^{-1})^{\frac{1}{2}}$, and $Q^\sharp$ its pseudo-inverse $\operatorname{diag}\{\sqrt{\frac{\lambda_1}{\lambda_1 - \lambda}}, \ldots, \sqrt{\frac{\lambda_{n-1}}{\lambda_{n-1} - \lambda}}, 0\}$. *$\mathfrak{R}^* = \mathfrak{B}(M^*)$ has flat angle* $\arcsin(\lambda^{\frac{1}{2}})$ *with respect to $\mathfrak{B}$, and its canonical gains equal those of $\mathfrak{B}$, with the smallest one removed.*

PROOF. According to Theorem 5.1, $\mathfrak{R}^*$ is given by $\mathfrak{B}(\mathbf{0} \wedge \mathbf{f}_n)^\perp = \mathfrak{B}(\mathbf{p}_n \wedge \mathbf{0})$, has degree $n - 1$, and indeed has flat angle $\arcsin(\lambda^{\frac{1}{2}})$ with respect to $\mathfrak{B}$. So (6.16) with $k = n$ and $r = 1$ is a state representation for $\mathfrak{R}^*$, with minimal state after removing the last (zero valued) component in $\mathbf{x}$. Observe that (A_R, B_R, C_R, D_R) is obtained by a basis transformation $Q^\sharp$ (which is invertible on the minimal state space) and a rescaling of auxiliary inputs by a factor $\widetilde{\lambda}^{\frac{1}{2}}$. This proves that $\mathfrak{B}(M^*) = \mathfrak{R}^*$. Further, the controllability gramian in (6.16) transforms into $Q^\sharp(W - \frac{\lambda_n}{\widetilde{\lambda}_n}\widetilde{W})Q^\sharp = \operatorname{diag}(\lambda_1, \ldots, \lambda_{n-1}, 0)$, which proves the claim on the canonical gains in the reduced system.

It remains to show that M^* is a unitary matrix. Straightforward verification turns out to be problematic, and instead we give a more abstract derivation of this fact, based on a further analysis of (6.16) for siso systems. If we take for $\widehat{\mathbf{v}}$ in (6.16) the value

$$\widehat{\mathbf{v}} = \widetilde{\gamma}_k \widehat{\mathbf{v}}_{(k)}, \tag{6.18}$$

as defined in (6.6), then $\mathbf{w}$ and $\mathbf{x}$ are given by

$$\mathbf{w} = (\mathbf{p}_{(k)} \wedge \mathbf{0}) \tag{6.19}$$

$$\mathbf{x} = (\widetilde{\gamma}_k^{-1} Q^{\frac{1}{2}} \widehat{\mathbf{x}}_{(k)}^{-} \wedge \mathbf{0}), \tag{6.20}$$

which can be proved as follows. Observe that $\widehat{\mathbf{v}}$, $\mathbf{x}$ and $\mathbf{w}$ in (6.16) are consistent with their definition in (6.11) and (6.12). Consequently, for the given value of $\widehat{\mathbf{v}}$, the corresponding $\mathbf{w}$ in (6.16) must satisfy $\mathbf{w} - \widetilde{\gamma}_k \widehat{\mathbf{w}}_{(k)} \in \widetilde{\mathfrak{B}}$, which holds true for (6.19). The existence of any other element $\mathbf{w}' \in \mathfrak{B}(\mathbf{0} \wedge \mathbf{f}_{(n)})^{\perp}$ with this property would imply that $\mathbf{0} \neq \mathbf{w} - \mathbf{w}' \in \widetilde{\mathfrak{B}} \cap \mathfrak{B}(\mathbf{0} \wedge \mathbf{f}_{(n)})^{\perp}$, which contradicts Proposition 4.8 (by assumption $\widetilde{\mathfrak{B}}$ has rank one). Hence (6.19) must be true. Substituting this formula in (6.11) yields (6.20), where it may be helpful to use that $\Sigma_{i=0}^{j-1} A^i FCA^{j-1-i} = A^j W - WA^j$ for all $j \in \mathbb{N}$.

Now it is easily verified that for the triple $(\widehat{\mathbf{v}}, \mathbf{x}, \mathbf{w})$ as specified above, and $\mathbf{z} := E_k^{\top} Q^{\sharp} \mathbf{x}$, $\mathbf{h} := \widetilde{\lambda}^{-\frac{1}{2}} \widehat{\mathbf{v}}$, it holds that for all $t \in \mathbb{Z}$, $\begin{pmatrix} \mathbf{z}_{t+1} \\ \mathbf{w}_t \end{pmatrix} = \begin{bmatrix} A_R & B_R \\ C_R & D_R \end{bmatrix} \begin{pmatrix} \mathbf{z}_t \\ \mathbf{h}_t \end{pmatrix}$, while also $\mathbf{h}_t^{\top} \mathbf{h}_t + \mathbf{z}_t^{\top} \mathbf{z}_t = \mathbf{z}_{t+1}^{\top} \mathbf{z}_{t+1} + \mathbf{w}_t^{\top} \mathbf{w}_t$. Conclude that $\begin{bmatrix} A_R & B_R \\ C_R & D_R \end{bmatrix}$ is isometric.

Finally, we have to show that the last column in M^* completes the others to a square unitary matrix. Orthogonality of the last column to the others in M^* is straightforward from the fact that M is a unitary matrix, which implies $C^{\top} D = -A^{\top} B$ and $C^{\top} \widetilde{D} = -A^{\top} \widetilde{B}$. Hence $\widetilde{B}_R^{\top} B_R + \widetilde{D}_R^{\top} D_R = e_n^{\top} A^{\top} Q Q^{\sharp} T B + e_n^{\top} C^{\top} (D - \beta \widetilde{D} e_n^{\top} B) = e_n A^{\top} T B - e_n^{\top} A^{\top} (B - \beta \widetilde{B} e_n^{\top} B) = 0$. Similarly, $\widetilde{B}_R^{\top} A_R + \widetilde{D}_R^{\top} C_R e_n^{\top} A^{\top} Q Q^{\sharp} T A Q + e_n^{\top} C^{\top} (C - \beta \widetilde{D} e_n^{\top} A) Q = e_n A^{\top} T A Q - e_n^{\top} A^{\top} (A - \beta \widetilde{B} e_n^{\top} A) Q + e_n^{\top} Q = 0$.

In order to derive that the last column has unit norm, we make use of the (typical siso) fact that

$$W^{-\frac{1}{2}} A W^{\frac{1}{2}} = S \widetilde{W}^{\frac{1}{2}} A^{\top} \widetilde{W}^{\frac{1}{2}} S \tag{6.21}$$

with S some diagonal sign matrix, i.e., with diagonal entries 1 or -1 and other entries zero. This is derived as follows. $W^{-\frac{1}{2}} A W^{\frac{1}{2}}$ represents the projection of $\sigma \mathfrak{B}^{\Leftrightarrow}$ onto $\mathfrak{B}^{\Leftrightarrow}$ and $\widetilde{W}^{\frac{1}{2}} A^{\top} \widetilde{W}^{\frac{1}{2}}$ represents the projection of $\widetilde{\mathfrak{B}}^{\Leftrightarrow}$ onto $\sigma \widetilde{\mathfrak{B}}^{\Leftrightarrow}$ with respect to the bases of $\mathfrak{B}^{\Leftrightarrow}$ and $\widetilde{\mathfrak{B}}^{\Leftrightarrow}$ as given in Proposition 4.7. If $m = 1$ and $q = 2$, then $\widetilde{\mathfrak{B}}^{\mathrm{rev}}$ has CISR $\left[\begin{smallmatrix} A & B & \widetilde{B} \\ JC & JD & J\widetilde{D} \end{smallmatrix}\right]$, with $J := \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$. Now (6.21) follows from an obvious relation between the past-future links in this system and $\mathfrak{B}^{\Leftrightarrow}$, and the fact that canonical past-future links are unique modulo a sign if all gains are distinct, cf. Proposition 4.7.

Then $\widetilde{B}_R^{\top} \widetilde{B}_R + \widetilde{D}_R^{\top} \widetilde{D}_R = \beta^2 \widetilde{\lambda} (e_n^{\top} A^{\top} Q^2 A e_n + e_n^{\top} C^{\top} C e_n)$. As $C^{\top} C = I_n - A^{\top} A$, this equals $\beta^2 \widetilde{\lambda} (1 - e_n^{\top} A^{\top} (I_n - Q^2) A e_n)$, and applying (6.21) twice results in

$$\beta^2 \widetilde{\lambda} (1 - e_n^{\top} S (W \widetilde{W})^{-\frac{1}{2}} A (W \widetilde{W})^{\frac{1}{2}} S \lambda W^{-1} S (W \widetilde{W})^{\frac{1}{2}} A^{\top} (W \widetilde{W})^{-\frac{1}{2}} S e_n) = \beta^2 (\widetilde{\lambda} - e_n^{\top} A \widetilde{W} A^{\top} e_n).$$

Substitute $A \widetilde{W} A^{\top} = \widetilde{W} - \widetilde{B} \widetilde{B}^{\top}$ and apply the definition of β to see that the squared norm is indeed 1. □

We remark that for systems where the weakest link is a *backward link*, the algorithm can be applied to to a CISR of $\widetilde{\mathfrak{B}}$, whose weakest link will be a weakest forward link, or, equivalently, to a CISR of $\mathfrak{B}^{\mathrm{rev}}$.

We conclude this section by a corollary that follows immediately from applying the above theorem several times.

Corollary 6.5 (Sequential Reduction)
Under the hypotheses of Theorem 5.1 (and also for the case $\lambda_n > 1 - \lambda_1$), k sequential optimal reductions of $\mathfrak{B}$ by one degree results in a $n-k$-th order system $\mathfrak{R}$ with $\arcsin(\lambda_{n-k+1}^{\frac{1}{2}}) \leq \theta(\mathfrak{R}, \mathfrak{B}) \leq \arcsin(\Sigma_{j=n-k+1}^{n} \tau_j)$, with $\{\tau_j\}_{j=1}^{n}$ the set of real numbers $\{\min(\sqrt{\lambda_j}, \sqrt{1-\lambda_j})\}_{j=1}^{n}$ in decreasing order.

7 Comparison with other methods

In this section we will compare some of the main results of the paper with earlier work. We start with a comparison to the Global Total Least Squares (GTLS) method, as introduced in [23, 24], and which is developed from the same behavioral perspective as the optimal angle approach. Since the analysis of this paper has many similarities with the balancing techniques for input-state-output systems (cf. [18]), it is clarifying to describe the connection in some detail. In Section 7.2 we relate our results to truncation by balancing, and in Section 7.3 to optimal Hankel-norm reduction. The detailed relation between Theorem 5.1 and optimal Hankel norm reductions is the topic of future research.

7.1 Comparison with GTLS

The GTLS method is designed for the construction of systems in $\mathbb{L}$ that optimally approximate, under the angle criterion, a given time series $\mathbf{w}$, either on finite time ([23]) or in ℓ_2^q ([24]). For $m, n' \in \mathbb{N}$, this amounts to finding a system $\mathfrak{B}^* \in \mathbb{L}$ of rank m and degree at most n' such that

$$\theta(\mathbf{w}, \mathfrak{B}^*) \tag{7.1}$$

is minimized. If $\mathbf{w}$ belongs to a system $\mathfrak{B} \in \mathbb{L}$ with rank m and degree n, then the difference with the problem of optimal angle approximation is that in GTLS just one trajectory is approximated, and not a whole system in worst-case sense. If $\mathbf{w}$ is a canonical past-future link in $\mathfrak{B}^{\Leftrightarrow}$ then we obtain the following result.

Proposition 7.1
For $\mathfrak{B} \in \mathbb{L}$ and let $\widehat{\mathbf{w}}_{(k)}$ denote its k-th canonical link, cf. Definition 4.6. Then $\mathfrak{B}(\mathbf{p}_{(k)} \wedge \mathbf{0})$ is a stationary point[1] *of the GTLS criterion (7.1) with $\mathbf{w} = \mathbf{w}_{(k)}$, $m = 1$ and $n' = n - 1$.*

Proof. Under projection onto $\mathfrak{B}(\mathbf{p}_{(k)} \wedge \mathbf{0})$, $\mathbf{w}_{(k)}$ falls apart into a seperate past and future, cf. Proposition 4.2. Consequently, the approximation and its state trajectory in $\mathfrak{B}(\mathbf{p}_{(k)} \wedge \mathbf{0})$ on the one hand, and the projection error and its state trajectory in $\mathfrak{B}(\mathbf{p}_{(k)} \wedge \mathbf{0})^{\perp}$ on the other hand, have zero covariance. This is precisely the stationarity condition, cf. [24, Theorem 8.1]. □

The question arises under which circumstances these systems are also *globally* optimal under the GTLS criterion. This is the case if $k = n = 1$ and $n' = 0$. Simulations do not exclude that it is also true for $n > 1$ and $k = n$, in case the weakest link in $\mathfrak{B}$ is a forward link. This is illustrated in the example in Section 8. This would imply that under these conditions optimal reduction by one degree under the angle criterion for $\mathfrak{B}$ is equivalent to the single-trajectory based GTLS criterion for $\widehat{\mathbf{w}}_{(n)} \in \mathfrak{B}$.

7.2 Comparison with balanced truncation

In this section we compare our results to the balanced truncation model reduction technique initiated by [18]. Since balanced truncation methods apply to stable input-output systems, the methods are best compared on the level of the (auxiliary) input-output mapping induced by the isometric state representations.

[1]More precisely, with the angle criterion as function of the parameters in a representation of the system, e.g. a CISR, this criterion has zero partial derivatives in the point in the parameter space corresponding to this system

For a CISR $M = \begin{bmatrix} A & B & \widetilde{B} \\ C & D & \widetilde{D} \end{bmatrix}$ let

$$G(z) := C(Iz - A)^{-1}B + D \tag{7.2}$$

and let $\mathbf{g}$ be the inverse z-transform of G. Associate with G an operator $G : \ell_2^m \to \ell_2^q$ defined by the convolution $\widehat{\mathbf{w}} = G(\widehat{\mathbf{v}}) = \mathbf{g} * \widehat{\mathbf{v}}$. Then, by construction, $\widehat{\mathbf{w}} \in \mathfrak{B}$ if and only if $\widehat{\mathbf{w}} = G(\widehat{\mathbf{v}})$ for some $\widehat{\mathbf{v}} \in \ell_2^m$. Here (A, B, C, D) is a realization of G, with observability gramian I_n and controllability gramian W, as defined in Theorem 6.1. A diagonal state transformation $W^{-1/4}$ brings (A, B, C, D) in balanced form in that the observability and controllability gramians are both equal to $W^{\frac{1}{2}}$. This implies that the (Hankel-)singular values of G are in one-to-one correspondance with the canonical gains. More precisely, the k-th singular value σ_k satisfies

$$\sigma_k^2 = \frac{1}{1 + \rho_k^2} (= \gamma_k^2 = \lambda_k),$$

with ρ_k the k-th canonical past-future ratio (cf. Definition 4.6). Consider the smallest singular value σ_n. It satisfies

$$\sigma_n^2 = \frac{\overrightarrow{\rho}}{1 + \overrightarrow{\rho}^2}$$

where $\overrightarrow{\rho}$ is the weakest forward gain of $\mathfrak{B}$, cf. Definition 4.4. If this is also the weakest gain, i.e, if $\overrightarrow{\rho} < \overleftarrow{\rho}$, and $m = 1$ or $m = q - 1$, then the optimal degree $n - 1$ approximant as defined in Definition 3.4 is obtained by cutting the weakest, or n-th canonical link. Truncation by balancing implies the annihilation of the n-th state component.

The two methods agree in the formalization of a concept of 'least important state', but they differ, however, in the implementation of the idea of annihilation of these states. In the method of balanced trunctions the rows and columns corresponding to the least important states are removed from the state representation. In the Matlab procedure `dmodred` these states are restricted to constant values and then eliminated. Both approaches turn out to be heuristic, in the sense that the approximate systems have no well-defined optimality properties.

The annihilation condition in the our approach is (6.14). The main point is that, on the one hand, the state $\mathbf{x}$ does coincide with the 'true' state $\widehat{\mathbf{x}}$ for system trajectories, cf. Section 6.4, so it may also be phrased as 'annihilating the last state component of the system'. On the other hand, $\mathbf{x}$ is defined more generally, for every $\mathbf{w} \in \ell_2^q$ (with value $\widehat{\mathbf{x}} + \widetilde{\mathbf{x}}$, cf. Section 6.4), with $\widetilde{\mathbf{v}}$ not necessarily zero. In fact, the proof of Theorem 6.4 shows that the state component is annihilated by $\widetilde{\mathbf{v}}$ alone, without adapting the value of $\widehat{\mathbf{v}}$. We consider it as a strong point of the behavioral approach to systems that optimal reduction seems to be much more straightforward on the level of 'balanced' *trajectories* (truncation of past-future links) than on the level of balanced *representations*.

7.3 Relation with Hankel-norm reduction

In view of the previous it is obvious that there must also be a close connection between OAR and Hankel approximations on the level of induced operators G. We assume that the system is of rank $m = 1$, so that G has a single input, and operators are defined by the image of one non-zero trajectory. The process of cutting the k-th canonical link suggests a decomposition

$$G = \widehat{G}_{(k)} + \widetilde{G}_{(k)}$$

with

$$\begin{aligned} G(\widehat{\mathbf{v}}_{(k)}) &= (\widetilde{\gamma}_k \mathbf{p}_{(k)} \wedge \gamma_k \mathbf{f}_{(k)}) & (7.3)\\ \widehat{G}_{(k)}(\widehat{\mathbf{v}}_{(k)}) &= (\widetilde{\gamma}_k \mathbf{p}_{(k)} \wedge \mathbf{0}) & (7.4)\\ \widetilde{G}_{(k)}(\widehat{\mathbf{v}}_{(k)}) &= (\mathbf{0} \wedge \gamma_k \mathbf{f}_{(k)}). & (7.5)\end{aligned}$$

It turns out that the stable part of $\widehat{G}_{(k)}$ is indeed the $k-1$-th order optimal Hankelnorm approximation of G. Namely, from Proposition 4.8 it follows that $\|\widetilde{G}_{(k)}\| = \gamma_k = \sigma_k$, while (6.18), (6.19) and Proposition 6.3 induce that $\widehat{G}_{(k)}$ has at most $k-1$ stable poles.

This means that for the induced operator G, *the optimal Hankel approximation of degree k is obtained by truncation of the future in the k-th canonical link, and then taking the stable part.*

The result shows that, despite the substantial difference in interpretation, from a technical viewpoint there is an immediate connection between optimal angle and Hankel approximation of isometric systems. One of the substantial differences remains that the angle criterion is time-symmetric, and optimal solutions may correspond to the first Schmidt pair, depending on the type of the weakest gain. Furthermore, Hankel approximation starts with a given input/output system and is usually not applied to the auxiliary operator G.

With some modifications the construction can also be applied onto (not necessarily isometric) input output system, which will be described in a seperate paper.

8 Simulation example

We illustrate the model reduction approach by a numerical example. For a second order system we determine its unique first order approximation under the angle-criterion. The example is chosen such that the approximation formulas remain reasonably simple. The exact numerical computationshave been performed in Mathematica.

Consider a system in two variables $\mathbf{u}$ and $\mathbf{y}$ described by

$$\mathbf{y}_t - \frac{1}{3}\mathbf{y}_{t-1} = \mathbf{u}_t - \mathbf{u}_{t-1} + \frac{1}{2}\mathbf{u}_{t-2}. \tag{8.1}$$

Formally, this defines the system $\mathfrak{B} = \{\mathbf{w} = \left(\begin{smallmatrix}\mathbf{u}\\ \mathbf{y}\end{smallmatrix}\right) \in \ell_2^2 \mid \text{(8.1) holds}\}$. If $\mathbf{y}$ is regarded as the output of $\mathbf{u}$, the system corresponds to the transfer function $\frac{6-6s^{-1}-3s^{-2}}{6-2s^{-1}}$, having poles $\{0, \frac{1}{3}\}$ and complex zeros $\{\frac{1}{2} \pm \frac{1}{2}i\}$, and realization

$$\begin{aligned} \mathbf{x}_{t+1} &= \begin{bmatrix} \frac{1}{3} & 0\\ 1 & 0\end{bmatrix}\mathbf{x}_t + \begin{bmatrix}1\\0\end{bmatrix}\mathbf{u}_t\\ \mathbf{y}_t &= \begin{bmatrix} -\frac{2}{3} & \frac{1}{2}\end{bmatrix}\mathbf{x}_t + \mathbf{u}_t. \end{aligned} \tag{8.2}$$

Using the algorithm in Section 6.3 this representation can be transformed to a canonical isometric state representation (6.1). In step 2 of the algorithm of Section 6.3 this gives $K = \begin{bmatrix} \frac{1}{4} & -\frac{1}{6}\\ -\frac{1}{6} & \frac{5}{36}\end{bmatrix}$, $S^\top S = K$ is solved for $S = \begin{bmatrix} 0 & \frac{1}{6}\\ \frac{1}{2} & -\frac{1}{3}\end{bmatrix}$, $R = \frac{2}{3}$ and $F = \begin{bmatrix} \frac{1}{3} & -\frac{2}{9}\end{bmatrix}$ and

$$\widehat{M} = \frac{1}{3}\begin{bmatrix} 2 & 1 & 0 & 2\\ -2 & 0 & 1 & 2\\ 0 & 2 & 2 & -1\\ 1 & -2 & 2 & 0\end{bmatrix}, \tag{8.3}$$

In the third step, we have to diagonalize the observability gramian $W = \frac{1}{595}\left[\begin{smallmatrix} 11 & -4 \\ -4 & 71 \end{smallmatrix}\right]$ by a unitary transformation U. A solution for U that also orders the diagonal elements is obtained via the SVD of W. It is given by

$$U = \begin{bmatrix} -\sqrt{\frac{1}{2}-\frac{15}{2\sqrt{229}}} & \sqrt{\frac{1}{2}+\frac{15}{2\sqrt{229}}} \\ \sqrt{\frac{1}{2}+\frac{15}{2\sqrt{229}}} & \sqrt{\frac{1}{2}-\frac{15}{2\sqrt{229}}} \end{bmatrix} \tag{8.4}$$

and it is unique modulo sign changes for each column. A canonical isometric state representation for $\mathfrak{B}$ is given by

$$M = \begin{bmatrix} \frac{1}{3}-\frac{14}{3\sqrt{229}} & -\frac{1}{2}-\frac{19}{6\sqrt{229}} & \frac{1}{3}\sqrt{\frac{1}{2}+\frac{15}{2\sqrt{229}}} & \frac{2}{3}\sqrt{1-\frac{2}{\sqrt{229}}} \\ \frac{1}{2}-\frac{19}{6\sqrt{229}} & \frac{1}{3}+\frac{14}{3\sqrt{229}} & \frac{1}{3}\sqrt{\frac{1}{2}-\frac{15}{2\sqrt{229}}} & \frac{2}{3}\sqrt{1+\frac{2}{\sqrt{229}}} \\ \frac{1}{3}\sqrt{2+\frac{30}{\sqrt{229}}} & \frac{1}{3}\sqrt{2-\frac{30}{\sqrt{229}}} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3}\sqrt{\frac{5}{2}+\frac{53}{2\sqrt{229}}} & \frac{1}{3}\sqrt{\frac{5}{2}-\frac{53}{2\sqrt{229}}} & \frac{2}{3} & 0 \end{bmatrix} \tag{8.5}$$

with gramian $W = \left[\begin{smallmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{smallmatrix}\right]$ with $\lambda_1 = \frac{41+2\sqrt{229}}{595} \approx 0.1198$ and $\lambda_2 = \frac{41-2\sqrt{229}}{595} \approx 0.0180$.

The canonical past-future ratios are given by $\rho_1 = \sqrt{(1-\lambda_1)/\lambda_1} = \frac{-7+\sqrt{229}}{3} \approx 2.7109$ and $\rho_2 = \sqrt{(1-\lambda_2)/\lambda_2} = \frac{7+\sqrt{229}}{3} \approx 7.3776$. The weakest backward gain of the system is given by $\overleftarrow{\rho} = \rho_1 \approx 2.7109$, the weakest forward gain by $\overrightarrow{\rho} = 1/\rho_2 \approx 0.1355$, which is also the weakest gain ρ. It has multiplicity one, and the corresponding weakest link is unique modulo scaling and sign.

The optimal approximation $\mathfrak{R}^*$ of degree one is generated by the past of the weakest (forward) link, or, equivalently, it is the orthogonal complement of the future of this link, which is the future effect of the second state variable. The corresponding CISR of $\mathfrak{R}^*$ is given by Theorem 6.4, and takes the value

$$M_{\text{red}} = \begin{bmatrix} \frac{11-\sqrt{229}}{18} & \frac{1}{9}\sqrt{\frac{53+5\sqrt{229}}{14}} & -\frac{2}{3}\sqrt{\frac{-2+\sqrt{229}}{7}} \\ \frac{7}{9}\sqrt{\frac{-13+11\sqrt{229}}{170}} & \frac{1}{9}\sqrt{\frac{51997-569\sqrt{229}}{1190}} & \frac{1}{3}\sqrt{\frac{2714-178\sqrt{229}}{595}} \\ -\frac{1}{3}\sqrt{\frac{-26+22\sqrt{229}}{85}} & \frac{2}{3}\sqrt{\frac{554+2\sqrt{229}}{595}} & 3\sqrt{\frac{41-2\sqrt{229}}{595}} \end{bmatrix} \tag{8.6}$$

If we eliminate the state variable, then the approximate system is described by

$$\mathbf{y}_t - \frac{73-13\sqrt{229}}{206}\mathbf{y}_{t-1} = \frac{274+2\sqrt{229}}{309}(\mathbf{u}_t - \frac{17-\sqrt{229}}{20}\mathbf{u}_{t-1}). \tag{8.7}$$

Hence, $\mathfrak{R}^* = \{\mathbf{w} = \left(\begin{smallmatrix} \mathbf{u} \\ \mathbf{y} \end{smallmatrix}\right) \in \ell_2^2 \mid \text{(8.7) holds}\}$ are the ℓ_2 solutions of (8.7). This is the unique first order system that has minimal angle with respect to $\mathfrak{B}$. This angle equals $\arcsin\sqrt{\frac{41-2\sqrt{229}}{595}}$, which is about 7.7 degrees. Any other first order linear time-invariant equation has a square summable solution that has larger angle with respect to square summable solutions of (8.1). Moreover, the angle is flat, which implies that every element of $\mathfrak{R}^*$ attains this angle with respect to $\mathfrak{B}$, and, conversely, every system trajectory in $\mathfrak{B}$ attains this angle with respect to $\mathfrak{R}^*$.

Finally we consider the result of sequential optimal reduction by one degree. Cutting the past-future link in $\mathfrak{R}^*$ results in the optimal static model $\mathfrak{R}^0$ defined by the system law

$$7\mathbf{u}_t = 6\mathbf{y}_t.$$

This system has a flat angle with respect to $\mathfrak{R}^*$, but the angle with respect to $\mathfrak{B}$ is not flat. According to Section 6.5, the latter angle is given by $\arcsin \nu$ with

$$\nu := \sqrt{\sup_{0\le\theta\le 2\pi} \|H^\top(e^{-i\theta})H(e^{i\theta})\|_\infty} \tag{8.8}$$

and $H(s) = \frac{7}{\sqrt{85}} G_u(s) + \frac{-6}{\sqrt{85}} G_y(s)$ with $G_u(s) = \frac{2s-6}{6s^2-9s-2}$ and $G_y(s) = \frac{-3s^2+6s-6}{6s^2-9s-2}$, which are the transfer functions from $\widehat{\mathbf{v}}$ to resp. the first and second component of $\widehat{\mathbf{w}}$ as induced by (8.5).

Straightforward exact computations show that $\nu = \frac{2\sqrt{229}}{7\sqrt{85}}$, which is achieved for $\theta = \arccos(\frac{6}{11})$. So $\theta(\mathfrak{B}, \mathfrak{R}^0) = \arcsin \nu \approx 27.967$ degrees. Remarkably enough, this turns out to be equal to the sum of the flat angles between the subsequent reductions, $\arcsin(\sqrt{\lambda_1}) + \arcsin(\sqrt{\lambda_2})$. To show that this is not an optimal static approximation, we also determine the angle between $\mathfrak{B}$ and the static system $\mathfrak{R}^1$ described by the difference equation $4\mathbf{u}_t = 5\mathbf{y}_t$, in a similar way. This angle is given by $\arcsin(\sqrt{\frac{143+2\sqrt{8530}}{2009}}) \approx 23.821$ degrees. Consequently, $\theta(\mathfrak{B}, \mathfrak{R}^1) < \theta(\mathfrak{B}, \mathfrak{R}^0)$, i.e., $\mathfrak{R}^0$ is not optimal.

9 Conclusions

We formalized an optimal model approximation problem in the behavioral setting for a class of linear time-invariant ℓ_2 models. A complete solution has been provided for systems of rank one and reductions of the degree of the to-be-approximated system with one. Reduced order models have been characterized as those models that can be realized by means of a completion process based on the weakest link trajectories of the system. Partial results on arbitrary degree reductions have been derived by sequential reductions. Algorithms have been given for the algebraic calculations of optimal approximate models, based on isometric state space representations of systems. The relation of the results to global total least squares algorithms, model reduction by balanced truncations and optimal Hankel norm reductions has been indicated. A simulation example is given and it is shown that an iterative scheme of sequential reductions is not optimal.

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